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Abstract

R&D intensity for small firms is high and persistent over time. At the same time, small firms are often financially constrained. This paper proposes a theoretical model that explains the coexistence of these two stylized facts. It is shown that self-financed R&D investment can distort the effort allocated to different projects in a firm. In a dynamic environment, it is optimal for the firm to invest in R&D projects despite the borrowing constraints. In addition, this paper shows that beyond a certain threshold, effort substitution between R&D and production appears. When transfers from investor to entrepreneur are large enough, R&D intensity decreases with respect to financial resources. Conditional on survival, the more innovative and financially constrained firms are, faster they grow and exhibit higher volatility.

Key Words: Moral Hazard, Endogenous Borrowing Constraints, Technological Change.
JEL Codes: 041,031,D86

1 Introduction

Apple, Dell and Google are examples of the wave of successful startups during the mid 1970s and 1990s and, currently, their revenues are comparable to the GDP of small countries such as Ecuador, Croatia, and Latvia[1]. They started as small firms with high intangible

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[1]Source: Fortune Magazine http://money.cnn.com/magazines/fortune/fortune500/2011/full_list/301_400.html and IMF data taken for 2011 Macroeconomic Outlook

investment and were successful in getting funding from outside investors. The following empirical facts for the United States show that the dynamics of a firm is intrinsically linked to the development of capital markets.

1. Small Firms Exhibit Higher R&D Intensity

R&D intensity is measured as the ratio between R&D investment and sales. Based on the Compustat database, Caves (1998) shows that R&D intensity was constant over time for a set of firms publicly traded for the period 1973-1986. This suggests that R&D intensity is independent of firm size (see Klette and Kortum (2004) for a survey). More recent studies, using an updated version of said database (1999–2007), show a downturn relationship between R&D intensity and firm size (see Akcigit (2009) and Park (2011)). For example, Akcigit (2009) estimates that a 10% rise in firm size (measured by sales) is associated with a 2.65% decrease in R&D intensity for the period 1980–2005.

Park (2011) identifies a common pattern in which small firms with high R&D intensity have significant growth through joint ventures. Park's study shows that in the early 1970s, small start-up firms did not have the means to invest in R&D. However, the 1980s exhibited rapid growth in joint ventures, start-up firms with zero revenue, and high R&D investment. Small firms found it easier to attract funding, technical support, and networking to facilitate their investment in R&D. It should be noted that this expansion of R&D appeared at a time when the financial system also expanded, and such situation not only made it easier for small firms to find funding, but also increased options for diversifying the risk associated with R&D investment.

2. The Most Innovative Firms are Often Financially Constrained.

Recent literature shows that firms with R&D intensity suffer from a lack of finance. Hall (2002) argues that this is so because the return on R&D investments is highly uncertain. Private information about the quality of projects creates a lemon problem between investors and entrepreneurs and the information gap drives a wedge between external and internal finance. Moreover, R&D activities are difficult to collateralise, which means that entrepreneurs may prefer to use internal resources to fund their R&D projects.

Other sources of funding, such as joint ventures, are highly volatile. Gompers and Lerner (2006) provide empirical evidence to support that volatility in the joint venture industry is associated with different trends in technological innovation. For example, during the economic boom between 1998 and 2000, funding was 30 times higher than in 1991. Investments were mainly made in internet (39%) and telecomunication technologies (17%). Subsequent technological revolutions generated investment opportunities that created volatility in the stock markets. It was very difficult for small and medium-sized firms to hedge against this risk which increased financial constraints.

At the aggregate level, there is evidence of a correlation between financial constraints and firm size distribution. Cabral and Mata (2003) study the distribution of firm size and the evolution of cohorts in Portuguese firms. They find that firm size distribution is skewed at the time of start-up, although its evolution over time follows a log-normal distribution. The authors also find that firms that are able to overcome financial constraints are more efficient and are able to determine their evolution in terms of firm size distribution.

As argued above, the development of the financial system is key to fostering technological growth. For instance, Gorodnichenko and Schnitzer (2010) show that financial frictions in developing economies affect innovation and firm's export activities. By using the BEEPS World Bank Survey, the authors found that financial constraint is negatively correlated to the degree of innovation in economies where the financial market is poorly developed.

3. R&D Expenditure is Stable Over Time and Increases for Small and Young Firms.

Brown et al. (2009) analyse R&D investment smoothing patterns arising from high adjustment costs (e.g., wages for highly-skilled workers and training costs). They find that, over time, R&D smoothing can be a response to higher adjustment costs when the sources of financial investment are highly variable (i.e., highly volatile cash and equity flows). They argue that firms use cash reserves to smooth R&D; particularly, young firms use cash holdings to reduce R&D volatility by about 75%. This occurred in the period of 1998–2002 when there was a consecutive boom and bust cycle in the United States' equity markets. They documented an upward trend in both cash flow and R&D expenditure from 1970 to 2006. From 1998 to 2002, equity issues and cash flow declined sharply, but R&D investment remained relatively constant. This suggests that a cash reserve acts as a buffer-stock that prevents dramatic variation in the firm's R&D investment.

This paper proposes a theoretical model that reconciles the empirical findings outlined above. A dynamic model is set up to includes technological shocks and moral hazard in the allocation of effort between standard production and R&D activities. A distinguishing feature of this approach is that the borrowing constraint is endogenous, as it is explained by technological risk. The model is based on an entrepreneur who is cash constrained and raises external funds from an outside investor. As the division of effort between standard production and R&D is non observable by the investor, the entrepreneur faces financial constraints.

The model resembles the relationship between innovative start-up and joint venture who provides financial resources to the entrepreneur once the first prototype is developed. The parties agree on the ownership of the project through shares and start the standardization phase. For example, in the 1970s, Apple Inc. started operations for the assembly-line production for Apple II. The project was established thanks to the partnership between Jobs, Wozniak, and venture capitalist, Mike Markkula. Google has a similar story as search engine projects were financial and technically supported, in their early stages, by the co-founder of Sun-Microsystem (Andreas Bechtolsheim).

This paper's framework is based on an entrepreneur with two types of projects: the first is standard production, the second is R&D. In each project, the entrepreneur exerts effort with certain degree of substitutability. Final production is a combination of the stock of intermediate goods and the level of entrepreneurial effort, while the growth rate of intermediate goods is determined by the R&D effort. Therefore, the effort allocated to standard production affects current cash flow, while R&D effort affects the value of the firm's equity.

An outside investor provides resources and is then repaid by the entrepreneur. The repayment is a share of the total output, such as company stock. Therefore, the investor's objective is to align incentives for effort provision by controlling the entrepreneur's cash flow. Here, the benchmark economy is characterised by efficient allocations under full information. A central planner maximises the aggregate surplus according to resource constraints and the law of motion for intermediate goods. The efficient contract entails that the effort allocated to each activity is independent of firm size. Hence, under full information, Gibrat's Law is satisfied [2]. As mentioned above, result is standard in the endogenous growth model and firm dynamics' literature, but it is not supported by recent empirical literature.

Under asymmetric information, there is a conflict of interest between the investor and the entrepreneur as the level of effort allocated to each activity is non-observable. The production of the final good is an imperfect measure of the level of effort allocated to standard production, and the entrepreneur is privately informed about R&D effort. Hence, the information asymmetry affects the investor's surplus as it has an impact on the expected present value of the entrepreneur's repayment. The investor faces a tradeoff between maximising current cash flow or maximising the value of the entrepreneur's equity.

[2] Gibrat's Law states that the firm's growth rate is independent of its size.

We find that the optimal contract leads to different allocations of financial resources, depending on whether it is optimal for the entrepreneur to invest in R&D or not. When it is optimal for the entrepreneur to invest in R&D, production effort increases and is concave with respect to finance provided by the investor. In turn, R&D effort decreases and is convex when there are more financial resources available. The investor uses the repayment and continuation value (future financial transfers) to reduce the misallocation of productive resources within the firm.

Several simulation exercises are implemented to study the sensitivity of the optimal contract. This paper evaluates the impact of falling productivity and increasing correlation between projects. In the first case, a fall in firm's productivity is associated with a reduction in both standard production and R&D intensity. This leads to tighter borrowing constraints, increased repayments made to the investor (in order to maintain incentives to provide effort to standard production), and reduced profit for the investor.

In the second case, increases in the degree of correlation between projects have a positive effect on R&D intensity which carries a spillover effect on standard production. The number of projects funded by the investor falls because the entrepreneur has a greater incentive to invest in R&D. In this case, the optimal repayment to the investor decreases and the positive impact on standard production leads to increases in the investor's profits.

This paper also contributes to the theory of endogenous growth models. Here, growth is driven by increasing the number of varieties, as in Romer's model (1996). The rate at which the number of varieties increases is given by the R&D effort, which is non observable by the investor. Technological risk is then introduced into the production of technology varieties to obtain a balanced growth path.

This paper analyses the impact of productivity and task substitutability shocks on the main statistical moments of the firm; in particular, expected growth and aggregate variance. In both cases, the driving force is R&D intensity. In the first one, when financial constraints are tight, the entrepreneur has a greater incentive to allocate effort to R&D, which has a positive impact on expected growth. This effect is reinforced by the impact of the optimal contract on the allocation of effort. There are two effects on aggregate variance. First, high R&D intensity is related to high firm volatility and financial resources have a negative overall impact on variance. Consequently, small firms are positively correlated with high variance and binding financial constraints.

The following sections will develop in detail the foregoing. To do so, this paper is structured as follows: Section 2 summarizes the supporting literature. The dynamic model and the optimal contract for the case studied herein are presented in Section 3. The implications of the optimal contract on the firm's dynamics are examined in Section 4. Finally, Section 5 presents the main conclusions.

2 Related Literature

This paper is mainly based on two strands of economic literature. The first strand concerns R&D investment, firm dynamics, and financial frictions. The second one considers recent literature on the relationship between dynamic contracts and borrowing constraints.

Financial frictions can affect R&D by creating barriers to entry. Aghion et al. (2007) provide an empirical study on how R&D frictions affect entry and post-entry growth of firms. They find that financial constraints matter for the entry of small firms. Therefore, a reduction in these barriers fosters growth because it allows small firms to take advantage of opportunities and enable reallocation of resources in favour of the most efficient firms. While the authors find that financial development also has a positive effect on post-entry growth of small firms, it has the reverse or null effect on large firms. In the model herein, financial frictions arise from asymmetric information problems. The investor does not observe how the entrepreneur allocates effort between production and R&D. Financial frictions come from the misallocation of internal resources between R&D and production.

However, this effect has an important cyclical component. Wälde and Woitek (2004) find that R&D investment for G7 countries tends to be pro-cyclical. Aghion et al. (2010) analyse the implications of volatility for short and long-term R&D investment. In periods of recession, the firm's earnings decline and so does its ability to finance R&D. Consequently, R&D investment has a greater effect on firms when they are financially constrained, which can amplify productivity and output. Moreover, there is evidence from OECD countries where R&D investments are more sensitive to liquidity shocks, and hence there is a negative correlation between volatility and growth.

Most studies analyse the implications of financial frictions for total factor productivity (TFP) dynamics. Although those papers do not study R&D decisions directly, they are useful as a benchmark to understand how financial frictions impact TFP and, therefore, firm dynamics. It is well known that R&D is an important component of TFP dynamics. Cooley and Quadrini (2001) study the dependence of firm dynamics on its size and age. They integrate persistent productivity shocks and financial frictions to replicate the main features of firm dynamics. Financial frictions are modelled as a premium on equity with respect to reinvesting profits. They also study the cost of debt default (i.e., costly state verification) and find that higher levels of debt are associated with higher volatility in the firm's profits.

Clementi and Hopenhayn (2006) setup a model of firm dynamics with financial frictions. In this model, market incompleteness is presented as a problem of asymmetric information between borrowers and lenders in an intertemporal setting. They study repercussions of borrowing constraints on firm's growth and survival. Borrowing constraints are modelled as a commitment problem that limits investment opportunities. The authors predict that while the conditional probability of survival would increase with the value of the firm's equity, the failure rate would decrease with firm size. Borrowing constraints are endogenous, while productivity dynamics are exogenous. Consequently, negative productivity shocks make borrowing constraints binding and this increase the cost of capital and decrease the growth of the firm.

Along the same lines, Midrigan and Xu (2010) study the extent to which financial frictions account for misallocation and, therefore, TFP losses. They find that TFP losses in emerging economies are around 5–7% in terms of output. This cost corresponds to the reallocation effect. In fact, as long as the firm accumulates internal funds, it is less constrained and becomes more efficient as it avoids large swings in productivity.

Moll (2010) shows how financial constraints are less binding when idiosyncratic productivity shocks are persistent. The study finds that the ability of entrepreneurs to accumulate internal resources depends largely on the persistence of productivity shocks over time. A firm that faces a sequence of positive productivity shocks accumulate more internal resources and relax borrowing constraints. TFP losses are associated with capital market imperfections, and they depend on the autocorrelation of idiosyncratic productivity shocks. They account for about 20% of productivity losses in developing countries.

The main characteristic of the model herein is that TFP is endogenous and persistent due to R&D investment; furthermore, the misallocation of effort between activities limits the ability to allocate internal resources to either standard production or R&D. As demonstrated by Buera and Shin (2013), reallocation is costly and can lead to higher fixed costs in the case of R&D investment. Their paper develops a model to analyse the implications of financial frictions on productivity. It studies a two-sector economy in which financial frictions distort capital accumulation and entrepreneurial talent. The main feature of this model is that establishments are differentiated in terms of fixed costs. Establishments in industries with large fixed costs are more dependent on external finance, and, therefore, more financially fragile. The authors argue that this mechanism explain differences in productivity between manufacturing and services in less-developed economies.

This paper also relates to recent literature on the relationship between dynamic contracts and borrowing constraints. DeMarzo and Sannikov (2006) study the relationship between the investor and the entrepreneur in a moral hazard environment. In this model, the entrepreneur has the incentive to divert part of his cash flow for his own benefit, and therefore, reduces the firm's mean cash flow. To align incentives, the investor must either control wages and reduce the funds allocated to the project or threat to terminate the contract prematurely. Given this setup, the optimal contract is for the entrepreneur to have a share of equity in the project and acquire a credit line so that, in case of failure, he is able to recover part of the funds invested in the project.

In the setup herein, there is also an incentive for the entrepreneur to reduce effort with respect to the full information case, and thus to reduce the cash flow offered to the investor. The main difference is that the diverted cash flow is allocated to productive activities for the firm. However, there is a trade-off because, even if there are incentives to reduce current cash flow, there is also an intertemporal effect due to the fact that R&D can increase future cash flow through its impact on growth and future equity. In this model, the optimal contract gives the investor an appropriate share of equity in the project so that he can take advantage of the firm's growth opportunities.

Another relevant article is Biais et al. (2007), which analyse large-scale risk prevention contracts. This paper studies an environment of moral hazard in which the individual exerts effort to prevent large losses. The main difference with respect to the earlier literature is that the principal could alter the size of the project. Thus, they have an additional tool for aligning incentives with the entrepreneur. The optimal contract shows that the investment depends on the entrepreneur's performance history. If, over time, the entrepreneur has accumulated a history of bad performance, the optimal strategy for the principal is to downsize.

In our model, the investor uses the repayment and the continuation value as tools to align incentives between activities. The optimal contract shows that, although there is a region where it is optimal for the entrepreneur to make both types of effort, the effect on

R&D decreases as the investor increases the continuation value. Therefore, the investor can alter the growth rate of the project using the continuation value.

DeMarzo et al. (2012) introduce a model in which financing constraints arise endogenously from moral hazard between the owner and the manager of the firm. The distinct characteristic of this framework is capital accumulation. The authors find a difference between marginal and average Q that is persistent over time. The main implication of such study is that investment is positively correlated with past profitability, past investment, and past managerial compensation. In the framework herein, the accumulation process is based on R&D investments that increase the variety of goods. The degree of financial slack depends largely on the complementarity of tasks. If activities complement each other, it is optimal for the investor to offer a repayment plan such that the entrepreneur is incentivised to exert effort in both activities. This will relax borrowing constraints and accelerate the growth of the firm.

3 The Model

This paper analyses R&D investment frictions in a growth environment. Hence, this section establishes a dynamic model of technological uncertainty. In order to characterise dynamic contracts, the argument herein follow the approach proposed by Sannikov (2008) and applied to the case of stochastic technological growth and multiple efforts. The first subsection describes the environment of the model, and the second subsection studies the first-best benchmark. Finally, optimal contracts under asymmetric information are also studied.

3.1 Technology, Profits, and Information[3]

Time is continuous. At each period of time t, there is an infinite lived entrepreneur that allocates effort to standard production and R&D task. R&D activities stochastically increase productivity. The entrepreneur contracts with an investor to obtain financial resources.

Technology

The entrepreneur produces a homogeneous good y_t. The entrepreneur uses as inputs a stock n_t of existing intermediate goods and their own effort e_1 like entrepreneurial effort. Formally:

[3]Through the document, for convenience, we use the notation for partial derivatives $f(x, y)$ with respect to x as $f_x(x, y)$

$$y_t = n_t \nu e_{1,t} + \varepsilon_t \tag{1}$$

where ε_t is stochastic disturbance such that $\varepsilon_t \sim N\left(0, \sigma^2\right)$ and ν is an exogenous productivity parameter that directly affects productivity in standard production task.

Here, complementarity between effort in standard production and the stock of intermediate goods is assumed. Through R&D activities, the entrepreneur can increase the stock of intermediate goods, which are accumulated by the following technology:

$$dn_t = n_t\left(\eta dq + dm\right) \tag{2}$$

where the parameter $\eta > 0$ measures the rate at which intermediate goods are accumulated over time. The accumulation of intermediate goods has two components: First, the firm can build up them according to the Poisson process $q\left(t\right)$. The entrepreneur chooses an R&D effort e_2 in order to increase n. Increments dq are determined by:

$$dq\left(t\right) = \begin{cases} 0 \text{ with probability} & 1 - e_2 dt \\ 1 \text{ with probability} & e_2 dt \end{cases}$$

The second component is the Poisson process dm, which characterises obsolescence. In particular, a firm that produces n units of goods faces a hazard rate μn of becoming a firm of size $n - 1$. The loss of goods is represented by:

$$dm\left(t\right) = \begin{cases} 0 \ \text{ with probability} & 1 - \mu dt \\ -1 \text{ with probability} & \mu dt \end{cases}$$

This specification implies that the mean technology growth rate is determined by R&D effort,

$$E\left(\frac{\dot{n}\left(\tau\right)}{n\left(\tau\right)}\right) = \eta e_2 - \mu$$

Information Structure and Strategies

The aim of this subsection is to characterise the information environment in which the entrepreneur and the investor interact. To do so, consider the set of technological shock stochastic processes,$\mathcal{Q}$; this set contains all possible technologies that are determined in turn by all the information generated by its path $\mathcal{Q} = \left\{q_t, \mathcal{F}_t; 0 \leq t \leq \tau\right\}$ defined in a space

$(\Omega, \mathbf{F}, \mathbf{P})$. Where, Ω is the sample size, $\mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}$ denotes the information set and $\mathbf{P}$ is the probability measure that follows a Poisson process with intensity e_2.

One strategy for the entrepreneur is to allocate effort to standard production and R&D $e_i \in E$ for $i = 1, 2$. The entrepreneur's effort is in itself a stochastic process $e_i = \{e_{i,t} \in E; 0 \leq t \leq \tau\}$ and measurable with respect to $\mathcal{F}_t$. This means that, based upon the path of $\mathcal{Q}$, it is possible to determine effort e_i.

Strategies for the investor are defined in terms of a repayment function in which there is also a stochastic process $\psi = \{\psi_t \in \Psi; 0 \leq t \leq \tau\}$ where ψ_t is determined by the observed output $\psi_t\left(y_j; 0 \leq j \leq t\right)$.

Profits

The entrepreneur is risk-neutral and the expected discounted profits are given by:

$$\pi^E = E\left[r \int_0^\tau \exp\left(-rs\right)\left[\left(\left(1-\psi_s\right) y_s - nc\left(e_{1,s}, e_{2,s}\right)\right) ds\right] + \exp\left(-r\tau\right) R\right] \tag{3}$$

where total production y_t is given by (1). ψ is the repayment from the entrepreneur to the investor. It is assumed that the repayment is a share of total output y, such as shares. $c\left(e_{1,t}, e_{2,t}\right)$ represents the unit cost effort for both activities, where $c(e_1, e_2)$ is defined as:

$$c\left(e_1, e_2\right) = \frac{1}{2}\left(e_1^2 + e_2^2\right) + \gamma e_1 e_2 \tag{4}$$

with $\gamma < 1$. When $\gamma = 0$, activities are independent; when γ is high, activities are highly correlated. Once the contract is terminated, the entrepreneur receives a payoff $R \geq 0$ from an external party.

The investor is risk-neutral, and derives profit from the discounted repayment it expects to receive from the entrepreneur. Note that the investor only observes aggregate output, which is an imperfect measure of the entrepreneur's level of effort.

The investor's profits are represented by:

$$\pi^I = E\left[r \int_0^\tau \exp\left(-rs\right)\left(\left(\psi_s y_s\right) ds\right) + \exp\left(-r\tau\right) L\right] \tag{5}$$

where L is the expected liquidation value of the project's assets. Therefore the contract specifies the repayment to investor ψ and the termination stopping time $\tau \geq 0$. If the business is liquidated, the investor receive a scrap value L.

3.2 The First-Best

This subsection studies the case in which the level of effort given to each activity is observable and verifiable by the investor. The social surplus is the expected discounted profits by the entrepreneur and investor. Therefore, the first-best allocation is the solution to the following problem:

$$V = \max_{e_1,e_2} E\left[r\int_0^\tau \exp\left(-rs\right)\left(y_s - nc\left(e_{1,s}, e_{2,s}\right)\right)ds + \exp\left(-r\tau\right)\left(R+L\right)\right]$$

subject to the law of motion of accumulation of goods (2). Using the Change Variable Formula for a Poisson process (see, Walde (2008)), the Bellman equation of this problem is :

$$rV\left(n\right) = \max_{e_1,e_2} r\left[y - nc\left(e_1, e_2\right)\right] + e_2\left[V\left(\left(1+\eta\right)n\right) - V(n)\right] - \mu n\left[V\left(n-1\right) - V(n)\right] \tag{6}$$

The first term represents current profit, while the second and third terms measure how the social surplus changes when there is a "technological jump". Consider the case of corner solutions: suppose first that$e_1 \neq 0, e_2 = 0$. In this case, the level of effort $e_1 = \bar{e}_1$ is constant for all $s < \tau$. It implies that the value function is given by $V^{st} = n\frac{v^2}{2}$. Note that in the other case $e_1 = 0, e_2 \neq 0$, there is no incentive to produce intermediate goods since the total production is zero. The contract termination is given by the following condition:

$$V^T = \min_{n^{min}}\left\{V^{st}, R+L\right\}$$

Where n^{min} is a threshold that determines the firm minimum scale level to operate in the market. Beyond that threshold the social return is $R+L$, the reservation value for the entrepreneur and the investor's liquidation value.

When there are interior solutions, the first order conditions are:

$$[e_1] \quad \nu = c_{e_{1,t}}\left(e_{1,t}, e_{2t}\right) \tag{7}$$

$$[e_2] \quad \frac{V\left(\left(1+\eta\right)n\right) - V\left(n\right)}{n} = c_{e_{2,t}}\left(e_{1,t}, e_{2t}\right) \tag{8}$$

First-order condition (7) means that the marginal product of providing effort in production must be equal to the marginal cost, which is constant. The second first-order condition shows that, at the margin, one unit of R&D effort must be equal to the marginal gain for the entrepreneur of increasing the stock of goods per unit of input. This marginal gain is measured as the difference between intertemporal profits when the entrepreneur invests in R&D and the case where there is no innovation. The following proposition shows the socially efficient contract under full information.

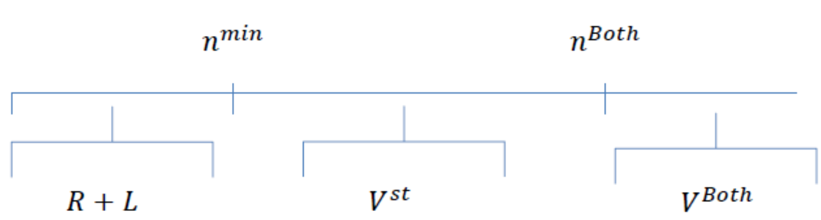
Proposition 1 : *If* $\{e_1^*, e_2^*\}$ *is the interior solution to the program (6), then* $V^{both} = f(e_1^*, e_2^*)\, n$ *where* $f(e_1^*, e_2^*) = \max_{e_1,e_2} \frac{r\left(\nu e_1^* - c\left(e_1^*, e_2^*\right)\right)}{r - \left(\eta e_2^* - \mu\right)}$ *and the Gibrat law holds (i.e R&D intensity is independent of the firm size).*

This proposition uses the property of homogeneity of degree one in the profits of the entrepreneur and the investor in such a way that the social surplus is linear with respect to the number of innovations. Based on this assumption and a frictionless environment, the Gibrat's Law is as in Klette and Kortum (2004). Therefore, there is a firm size threshold that determines the incentives for the firm to provide effort in both activities n^c which gives the value of continuation in the contract:

$$V^C = \max_{n^c} \left\{V^{st}, V^{both}\right\}$$

Figure 1 shows different regions in which, depending on the innovation size, the entrepreneur has incentives to allocate effort in standard production, both activities or just settle the contract.

Figure 1: Optimal Stopping



From Proposition 1, it is known that R&D intensity and effort are independent of firm size. However, How does the allocation of effort between activities distort the intertemporal margins? Take as an example the cost function (4); subtract (7) and (8) to get:

$$\frac{V((1+\eta)\,n) - V(n) - (1-\psi_t)}{n} = (1+\gamma)\,(e_2 - e_1)$$

Therefore, the efficient allocation of effort given to each activity depends on the difference between the marginal gain of the innovation and the share of output that is consumed by the entrepreneur. If R&D is profitable, then R&D effort efficiency is greater than the efficient level of production. This result is standard in a Schumpeterian endogenous growth model where there is over-investment in R&D.

The decentralised decisions are characterised by the following timing:

Timing

At time 0

- The investor proposes a contract $[y_t \mapsto \psi_t]_{0\leq t\leq\tau}$ to the entrepreneur.
- The entrepreneur either accepts or refuses the contract. If the contract is rejected, the game ends and there is no production.
- The entrepreneur establishes a path of effort in production and R&D $[e_1, e_2]_{0\leq t\leq\tau}$

At each t

- The level of output is realized y.
- The entrepreneur makes a transfer to the investor according to the level of output $\psi\,(y)$.

The Equilibrium

An equilibrium in this economy is a collection of stochastic processes $\{y_t, \psi_t, e_{1,t}, e_{2,t}\}_{t\geq 0}$ that are $\mathcal{F}_t$ - adapted and r such that in each time period t:

- Given r and ψ, the entrepreneur maximizes his profits.
- The monopolistic investor chooses the repayment ψ such that they maximize their profits.

and limited liability condition:

- $\psi(y) \leq y$

The following subsections consider the case in which the entrepreneur holds private information about the level of effort. This is done by characterising the optimal decisions of each agent and the optimal contract.

3.3 Moral Hazard

Now, consider the case in which the level of effort given to each activity is non observable. Therefore, the problem solved by the investor is to choose the repayment such that it implements levels of effort that are incentive-compatible with the entrepreneur's decisions, in such a way that the entrepreneur's discounted profits are maximised. This can be described as follows:

$$\pi^I = \max_{\psi_t, \tau} E\left[r \int_0^\tau \exp(-rs)\left((\psi_s y_s)\, ds\right) + \exp(-r\tau)\, L\right]$$

subject to the incentive compatibility constraint:

$$\left(e^{sb}_{1,s}, e^{sb}_{2,s}\right) = \arg\max_{\hat{e}_{1,s}, \hat{e}_{2,s}} E\left[r \int_0^\tau \exp(-rs)\left[\left((1-\psi_s)\, y_s - nc\left(\hat{e}_{1,s}, \hat{e}_{2,s}\right)\right) ds\right] + \exp(-r\tau)\, R\right]$$

and the promise-keeping condition in which the investor delivers to the entrepreneur a certain level of profit in order to incentivise them to participate in the contract.

$$E\left[r \int_0^\tau \exp(-rs)\left[\left((1-\psi_s)\, y_s - nc\left(e_{1,s}, e_{2,s}\right)\right) ds\right] + \exp(-r\tau)\, R\right] \geq \widehat{W} \tag{9}$$

The approaches by Spear and Srivastava (1987), Abreu et al. (1986), and more recently, Sannikov (2008) are here used to analyse the optimal contract. The main idea is for the investor to use the continuation value as a contractual instrument to keep track of incentives for the entrepreneur. Therefore, the contract wording follows the continuation value, W_t, which acts as a state variable for the investor.

Optimal contract allocations depend on historical information since the investor needs to track the whole history of technological shock in order to infer information on the agent's effort. Under the promised entrepreneur profit framework, agent's effort represents the main statistics of the contract, such as the level of effort given to each activity, the output in each period, transfers from the investor to the entrepreneur, and variations with respect to the different realizations technology takes. The investor uses the continuation value W_{t+dt} as a control variable, takes W_t as given and turns W_t into a new endogenous state variable. In this way, the problem can be written in a recursive form and standard dynamic programming techniques can be applied.

Following the Sannikov (2008) procedure, the first step is to study the continuation value as a diffusion process. This allows the carachterisation of the dynamics of the promised entrepreneur profit. The second step is to establish the conditions for incentive-compatible allocation of efforts. Third, using the previous input, the problem of the investor can then be written in recursive form and study the intertemporal conditions of the optimal contract.

3.3.1 The Continuation Value of the Entrepreneur

The contract specifies a flow of repayments $\{\psi_t, 0 < t < \infty\}$, and a sequence of efforts, $\{e_{1,t}, e_{2,t}; 0 < t < \infty\}$. The continuation value of the entrepreneur's profits is defined by adopting $\hat{e}_1, \hat{e_2}$:

$$W_t = E_{\hat{e}_{1,t},\hat{e}_{2,t}}\left[r\int_t^\tau \exp\left(-r\left(s-t\right)\right)\left[\left(\left(1-\psi_s\right)y_s - n_s c\left(e_{1,s}, e_{2,s}\right)\right)dt\right] + \exp\left(-r\left(\tau-t\right)\right)R \mid \mathcal{F}_s\right] \tag{10}$$

The standard techniques of the martingale representation theorem are used to characterise the dynamics of the promised entrepreneur profit. This equivalence theorem states that every martingale can be represented by an alternative process. In the context herein, this theorem is extended to a Poisson process (see Biais et al. (2010) and Bjork (2011)). It is first shown that the expected discounted profit of the entrepreneur is a martingale and then, it can be represented by an alternative process. Based on this equivalence, the expected profit is expressed as a function of the promised profit, and therefore an expression of the evolution of the continuation profit is obtained. The next two results show the application to the problem herein.

Lemma 1 : π_t^E *defined by*

$$\pi_t^E = r\int_0^t e^{-rx}\left[\left(\left(1-\psi_x\right)y_x - nc\left(\hat{e}_{1,x}, \hat{e}_{2,x}\right)\right)dx\right] + e^{-rt}W_t$$

is a $\mathcal{F}_t$*- adapted martingale.*

By using this result, it is now possible to characterise the continuation value as a diffusion process:

Proposition 2 : *Let q and m be two independent Poisson process and let $p = q+m$ be also a Poisson process with intensity $e_2 - \mu$ that admits the following martingale representation:*

$$dN_s = dp - (e_{2,s} - \mu)\, ds \tag{11}$$

then there exists a measurable process $h = \{h_t, \mathcal{F}_t; 0 \leq t \leq \infty\}$ such that:

$$dW_t = r \left\{ \underbrace{W_t - [(1-\psi_t)\, y_t - nc\,(e_{1,t}, e_{2,t})]}_{Cash\ Reserves} - \underbrace{W h_t\,(e_{2,t} - \mu)}_{R\&D\ Deviation} \right\} dt + \underbrace{rW_t h_t dp}_{Growth\ of\ varieties} \quad . \tag{12}$$

The continuation value grows with the entrepreneur's discounted rate and decreases with his net current profits. The technological innovation affects the dynamics of the continuation utility in two ways. First, there is a negative effect that comes from the effort given to R&D, $h_t e_{2,t}$. Alternatively, there is a positive effect as more innovations increase the number of goods. This is captured by $\frac{1}{\eta}(dn_t/n_t) = dq$, where the stochastic component is given by the amount of goods that the firm accumulates .

Given the structure of the cost function, R&D effort affects not only the creation of goods in the economy, but also generates a 'crowding-out' effect with respect to production effort. Therefore, as R&D effort raises, the arrival rate of new goods increases, while the level of effort given to current production decreases. The factor h_t measures the responsiveness of the continuation value to technological uncertainty. As Biais et al. (2010) explain, this factor represents the "minimum penalty" that provides an incentive to the agent to exert R&D effort.

Note that the drift is composed of two parts: the first refers to the cash reserves which are equivalent to the difference between the amount of assets owned by the entrepreneur minus the amount of resources saved from the production process. The second part corresponds to the amount of resources invested by the entrepreneur in the R&D projects. The continuation profit is also affected by an stochastic component that captures the variety growth.

3.3.2 Incentive Compatibility

Based on changes in the continuation value, it is possible to characterise the best response effort from the entrepreneur. That means that it is possible to measure how the continuation value varies with each level of effort. Allocations that are incentive compatible imply two conditions: first, they require local incentive constraints that state that the entrepreneur receives more profits when exerting higher effort if compared to the case in which he only exerts high effort in one of the two activities:

$$E_{e_{1,t},e_{2,t}} \int_0^{\tau} exp(-rs)n_s \left[\left(\left(1-\psi_s\right)\nu e_{1,s} - c\left(e_{1,s}, e_{2,s}\right)\right) ds\right] \geq E_{e_{1,t},\tilde{e}_{2t}} \int_0^{\tau} exp(-rs)n_s \left[\left(\left(1-\psi_s\right)\nu e_{1,s} - c\left(e_{1,s}, \tilde{e}_{2,s}\right)\right) ds\right]$$

and

$$E_{e_{1,t},e_{2,t}} \int_0^{\tau} exp(-rs)n_s \left[\left(\left(1-\psi_s\right)\nu e_{1,s} - c\left(e_{1,s}, e_{2,s}\right)\right) ds\right] \geq E_{\tilde{e}_{1,t},e_{2,t}} \int_0^{\tau} exp(-rs)n_s \left[\left(\left(1-\psi_s\right)\nu \tilde{e}_{1,s} - c\left(\tilde{e}_{1,s}, e_{2,s}\right)\right) ds\right]$$

Secondly, these allocations require a global incentive constraint under which the entrepreneur's profits from providing high effort in both activities are higher than if no effort had been exerted in either of them:

$$E_{e_{1,t},e_{2,t}} \int_0^{\tau} exp(-rs)n_s \left[\left(\left(1-\psi_s\right)\nu e_{1,s} - c\left(e_{1,s}, e_{2,s}\right)\right) ds\right] \geq 0$$

The following proposition characterises the incentive compatibility constraint:

Proposition 3: *The pair of effort (e_1, e_2) is incentive compatible if and only if :*

$$\frac{h_s}{r} e_{2,s} W_s - n_s c\left(e_{1,s}, e_{2,s}\right) \geq \frac{h_s}{r} \tilde{e}_{2,s} W_s - nc\left(e_{1,s}, \tilde{e}_{2,s}\right) \tag{13}$$

and

$$\left(1-\psi_s\right)\nu e_{1,s} - c\left(e_{1,s}, e_{2,s}\right) \geq \left(1-\psi_s\right)\nu \tilde{e}_{1,s} - c\left(\tilde{e}_{1,s}, e_{2,s}\right) \tag{14}$$

for each s.

Proposition 3 shows that incentive compatibility includes not only current profit but also the sensitivity of R&D intensity to the continuation value of the entrepreneur's profit. Therefore, a strategy (e_1, e_2) is optimal if, and only if, in each time period the agent maximises current profits and the expected impact of R&D intensity on the continuation value.

Incentive constraint [13] shows that the expected gain for an entrepreneur following strategy e_2 is higher than for other strategies. This incentive constraint is affected by the sensitivity factor that measures the impact of the frequency of innovation on profits. The second incentive constraint measures the production margin. The net profit from exerting high production effort is greater than exerting low effort in every time period. The following expression is the result of adding these two constraints together:

$$\frac{h_s W_s}{n_s r}\left(e_{2,s} - \tilde{e}_{2,s}\right) + \left(1 - \psi_s\right)\nu\left(e_{1,s} - \tilde{e}_{1,s}\right) \geq c\left(\tilde{e}_{1,s}, e_{2,s}\right) - c\left(e_{1,s}, \tilde{e}_{2,s}\right) \tag{15}$$

The aggregate incentive constraint shows that the expected gain for the entrepreneur from production and R&D is higher that the opportunity cost of performing only one of the tasks. As shown by Sannikov (2008), the set of incentive compatibility constraints satisfies:

$$h_s = n_s r c_{e_{2,s}}\left(e_{1,s}, e_{2,s}\right)$$

and

$$\left(1 - \psi_s\right)\nu = c_{e_{1,s}}\left(e_{1,s}, e_{2,s}\right) \tag{16}$$

3.3.3 Recursive Representation and Optimal Contract

This subsection describes in recursive form the investor strategy outlined in Section 3.2 . Based on previous results, the continuation value is used as a state variable for the investor. The characterization of the evolution of the state variable shows that it only depends on current variables. The investor's objective is to obtain the highest profit $\pi^I(W)$ while delivering a level of W to the agent. Consequently, the investor's problem can be expressed as current profits plus expected discounted profits $\frac{1}{dt}E_t d\pi^I(W)$. By using equation [12] and the Change Variable Formula for a Poisson process, we compute [12] as follows:

$$Ed\pi^I\left(W_t, n_t\right) = r\pi_w^I\left[W - (1-\psi)\,y + nc\left(e_1, e_2\right) - Whe_2\right]dt + e_2\left[\pi^I\left(\widetilde{W}\right) - \pi^I(W)\right]$$

Therefore, the Bellman equation for the investor is given by

$$r\pi^{I}(W,n)=\max_{e_1,e_2,\psi} r(\psi y)+r\pi_w^{I}\left[W-(1-\psi)\,y+nc(e_1,e_2)-Whe_2\right]+e_2\left[\pi^{I}\left(\widetilde{W}\right)-\pi^{I}(W)\right] \tag{17}$$

where $\widetilde{W} = W(1+rh)$. Notice that the jump is endogenous and it depends on the marginal impact of continuation value in the contract measures by Wh. Therefore, as examined below, the investor can alter the entrepreneur incentives through the financial resources that in turn affect the R&D intensity. The model has the property of a constant return to scale, therefore:

$$\pi^{I}(W,n)=n\pi^{I}\left(\frac{W}{n},1\right)=n\pi^{I}(z)$$

where $z=\frac{W}{n}$. Bellman equation is:

$$r\pi^{I}(z)=\max_{e_1,e_2,\psi} r(\psi e_1\nu)+r\pi_z\left[z-(1-\psi)\,e_1\nu+c(e_1,e_2)-zhe_2\right]+e_2\left[\pi^{I}(\widetilde{z})-\pi^{I}(z)\right] \tag{18}$$

with the following boundary conditions:

$$\pi_z(\bar{z})=-1,\pi_{zz}(\bar{z})=0$$

where $\bar{z}$ upper bound of state z

The first term in Bellman's equation corresponds to the current reward (net repayment). The second term captures the expected discounted value of the profit, which in turn has been decomposed into drift in the entrepreneur's continuation value and the effect of the technological jump on the investor's profits. The drift in the entrepreneur's continuation value impacts the marginal value of the investor profits, which is affected by the difference between the delivered and current profits. If the entrepreneur's current profits are high enough, the impact of delivering one unit of continuation value to the entrepreneur decreases the investor's profits.

R&D effort affects the investor's profits in two ways. Firstly, it can negatively affect the investor's marginal profits since it reduces the provision of effort in standard production for the entrepreneur. Secondly, the positive effect is that it increases the investor's profits when there is a technological jump.

The optimal contract is characterized by studying the case in which allocations are incentive compatible, i.e., when the entrepreneur has incentives to exert high effort in both activities e_1^*, e_2^*. Hence, the optimal contract analyses how much of the entrepreneur's continuation profit the investor delivers to the entrepreneur given that the entrepreneur provides high effort and participates in the contract. The optimal contract implies that there is a trigger strategy where there are several thresholds of continuation profit in which the entrepreneur is incentivised to exert high effort in only one, or both activities.

Let $\pi^{prod}(z), \pi(z),$ *and* $\pi^{R\&D}(z)$ be the level of investor profits, in the cases when the entrepreneur provides either only production effort, effort to both activities, or only R&D effort. The following two propositions characterise the optimal contract:

Proposition 4: *Consider the following parameters conditions:* $\gamma \in [0,1]$, $1+\frac{1}{\gamma^2} > r\left(1+2r^2\right)$ *and* $v \geq \gamma^2$. *The optimal contract is a set of* $e_i, i = 1,2$, ψ *and continuation values* $z = \frac{W}{n}$, *such that these are solution to [17] and lies on* $z \in [0,1]$, *there is a threshold* $\hat{z}$ *that satisfies:*

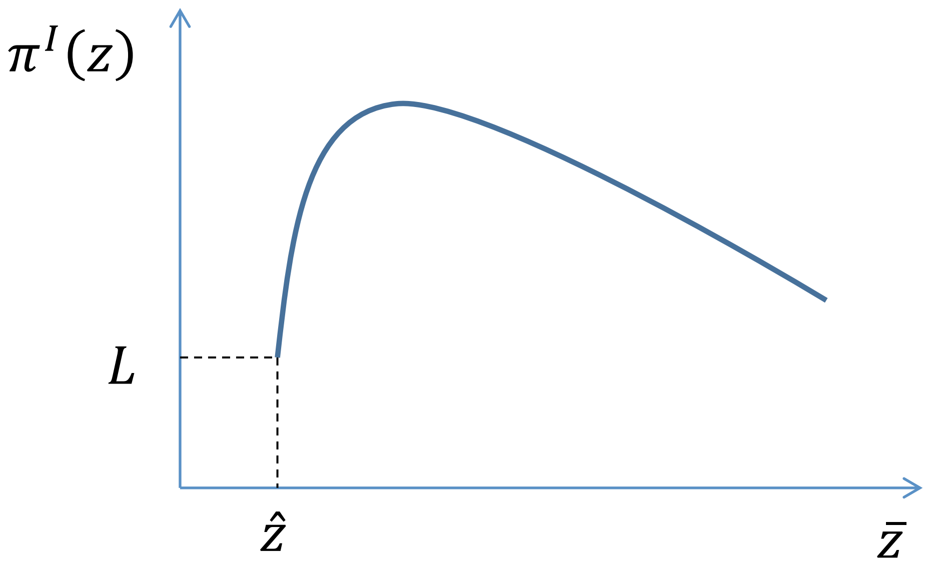
if $\;z > \hat{z}\;$ *then* $\;e_1^{sb}$, *is increasing and concave with respect to* z *in* $[\hat{z}, \bar{z}]$, e_2^{sb}, ψ^{sb} are decreasing and convex *with respect to* z *in* $[\hat{z}, \bar{z}]$.

In all other cases, investment only in production or in R&D is suboptimal means $\pi^{prod}(z) < \pi(z)$, $\pi^{R\&D}(z) < \pi(z)$. *The thresholds are determined by the indifference points.*

Proposition 5 : Let be $\pi(z)$ *a continuous and differentiable function in* $(\hat{z}, \bar{z})$, $\pi(z)$ *is concave with respect to* z *in* $[\hat{z}, \bar{z}]$.

In the interval $[0, \hat{z})$, the investor obtains negative profits. Therefore $[0, \hat{z})$ is an inaction region that determines the free-entry condition of the problem. The relevant region for the entrepreneur is $(\hat{z}, \bar{z}]$ where effort is exerted in both activities. As stated in Proposition 5, the investor's profits are concave. As Figure 2 shows, initially, profit must cover the scrap value. The shape of the curve depends on the magnitude of the misallocation between R&D and production..

Figure 2: Investor´s Value Function

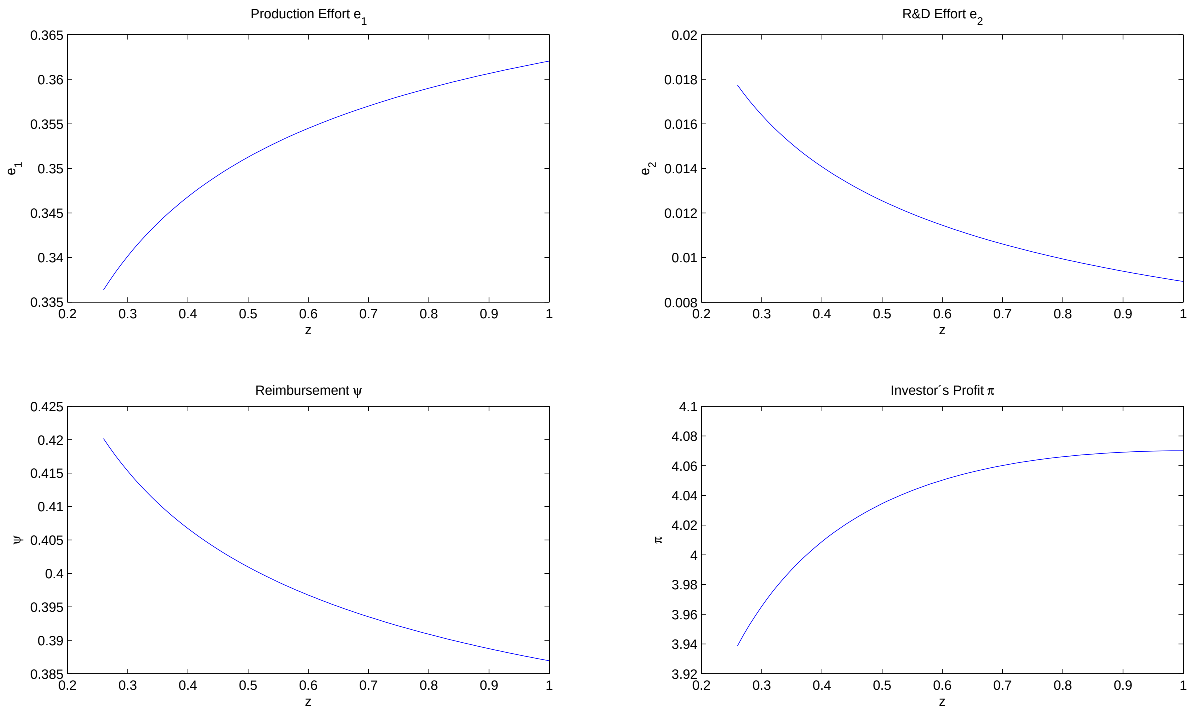


In Figure 3 a standard optimal contract has been parametrised. The optimal contract requires that the entrepreneur provides a monotonically increasing level of effort in production but there are marginal decreasing returns as the continuation value increases. R&D effort has the opposite effect, i.e., low continuation values are associated with high R&D intensity. The degree of misallocation increases as the continuation value rises.

The optimal repayment imposed by the investor captures potential changes in effort levels of the entrepreneur. It implies that the investor creates incentives over time, assigning rents to the entrepreneur to generate high levels of effort. Furthermore, as repayments decrease, the entrepreneur invests more resources in R&D. In figure 3, beyond a certain value of $\hat{z}$, the entrepreneur has the resources to increase their innovative activity. However, the optimal contract shows that substitution effects between activities lead to a decrease in R&D intensity over time.

When the investor is interested in the effort allocated in both activities, the continuation value will affect not only the entrepreneur's current profits, but also their future profits through R&D investment. The continuation value will affect the sensitivity factor that alters R&D effort and, therefore, the entrepreneur's profit. The sensitivity factor captures the opportunity cost for the entrepreneur of exerting R&D effort (or not doing so).

Figure 3: Typical Optimal Contract: $r = 0.04$;$\gamma = 0.95$; $\nu = 0.90$



4 Firm Dynamics

This section analyses the impact of financial frictions on firm dynamics based on the framework developed in Klette and Kortum (2004). In particular, the potential impact of misallocation on the firm's growth rate and variance.

The evolution of the firm is given through the dynamics of the stock of goods. The investor's financing decisions affect the firm's growth rate due to its impact on R&D intensity. For now, it is considered that an entrepreneur takes as given the financial resources provided by the investor. The firm changes the stock of goods over a time interval in accordance with the following continuous time Markov chain:

1. $\Pr\left(z(t+\triangle t) - z(t) = z + k \mid z(t)\right) = z\eta e_2\left(z\right)\triangle t + o\left(\triangle t\right)$
2. $\Pr\left(z(t+\triangle t) - z(t) = z - k \mid z(t)\right) = z\mu\triangle t + o\left(\triangle t\right)$
3. $\Pr\left(z(t+\triangle t) - z(t) > z + k \mid z(t)\right) = o\left(\triangle t\right)$

4. $\Pr\left(z(t+\triangle t)-z(t)=0 \mid z(t)=i\right)=1-z\left(\eta e_2-\mu\right)\triangle t+o\left(\triangle t\right)$

The forward-looking Kolmogorov differential equation is computed to characterise the life cycle of the firm over time. Said equation describes the probability distribution of firm size over time:

$$p_z\left(t+\triangle t \mid z_0\right)=\begin{array}{c}\left(z-k\right)p_{z-k}\left(t \mid z_0\right)e_2\triangle t+\left(z+k\right)p_{z+k}\left(t \mid z_0\right)\mu\triangle t \\ zp_z\left(t \mid z_0\right)\left(1-\left(\eta e_2-\mu\right)\triangle t\right)+o\left(\triangle t\right)\end{array}+$$

Taking the limit when $\triangle t \to 0$ the following expression is obtained:

$$\dot{p}_z\left(t \mid z_0\right)=\begin{array}{c}\left(z-k\right)p_{z-1}\left(t \mid z_0\right)e_2+\left(z+k\right)p_{z+k}\left(t \mid z_0\right)\mu \\ zp_z\left(t \mid z_0\right)\left(1-\left(\eta e_2-\mu\right)\right)\end{array}+ \tag{19}$$

A firm of size $z-k$ grows to size z with probability $\left(z-k\right)e_2$. There is a probability $n\mu$ that a firm with size $z+k$ downsizes from z to $z-k$. Similarly, the probability that a firm does not innovate and remains the same size is given by $z\left(1-\left(e_2-\mu\right)\right)$. The following lemma presents the main statistical moments for the firm:

Lemma 2 (Klette and Kortum (2004)): *Given z, the expected rate of firm's growth and variance are given by:*

$$g\left(z\right)=\max\left\{0,\exp\left[\left(e_2\left(z\right)-\mu\right)t\right]-1\right\} \tag{20}$$

$$var\left(z\right)=\frac{e_2\left(z\right)+\mu}{e_2\left(z\right)-\mu}\exp\left[\left(e_2\left(z\right)-\mu\right)t\right]\left[\exp\left(e_2\left(z\right)-\mu\right)t-1\right] \tag{21}$$

Now, it is possible to relate the dynamics of the firm with access to credit using the properties of the optimal contract. In the model, the measure of firm's financial constraints is$\mid \hat{z}-\bar{z}\mid$. The question that arises is how does substitution between standard production and R&D affect the firm's growth rate? The answer is found by studying the dynamics of the growth rate over time. Start by assuming that the firm's size follows an exponential distribution. This assumption is reasonable in the sense that, in the long run, firms tend to disappear.

According to the optimal contract, beyond $\hat{z}$, the firm does not grow because the level of R&D intensity is zero. $\overline{z}$ defines the maximum resources provided by the investor and is taken exogenous. The following proposition studies the firm's growth and variance at different stages of financing.

Proposition 6: *Conditional on survival, a firm that it is more financially constrained grows, on average, faster and exhibits higher variance.*

The proposition shows that the threshold $\hat{z}$ changes with variations in the investor's continuation value z. Changes in this threshold determine the entrepreneur's incentives. Increasing the threshold generates incentives for the entrepreneur to assign more resources to R&D than standard production. This has a positive effect on the average growth rate because higher R&D has a positive effect when financing is scarce.

On the other hand, changes in z have two impacts on variance: $var_z(z) = var_{e_2}(z)\, e_{2,z}(z)$. As it is shown in proposition 6, the R&D intensity is positive related to the individual variance. The fact that R&D is a risky activity is reflected in sales variability. Second, based on the optimal contract, when there are more financial resources, the final effect is amplified. The model predicts that small innovative firms face more cash flow volatility that makes it more difficult to find financing from an outside investor.

4.1 Numerical Exercises and Comparative Statics

This subsection discusses the sensitivity of the optimal contract to negative productivity shocks ν and changes in the substitutability parameter γ; in particular, how idiosyncratic shocks affect the firm's borrowing constraints. Parameters are based on Compustat data at the level of firm for the United-States in the period of 1980–2007. Table 1 shows the main parameters used in the simulations which are set according to the following first order conditions in steady-state to calibrate an average economy (see table 2). For the purpose of simplicity, fix $z^{ss} = 1$ value at the steady-state which implies $W^{ss} = n^{ss}$. The substitutability parameter γ and the reinbursement rate are calibrated using the steady-state values of effort in production and R&D:

$$\gamma = \left(1 - \frac{v}{e_1^{ss}}\right)^{1/2}$$

$$\psi^{ss} = 1 - \frac{(e_1^{ss} + \gamma e_2^{ss})}{\nu}$$

Table 1: Benchmark Parameters

γ	ν	r	ψ^{ss}
0.95	0.90	0.04	0.59

Table 2 presents average values for a representative firm for the following items: production effort (e_1), R&D intensity (e_2), repayments (ψ), growth $E\left(g^A\right)$ and variance var^A. The 'Data' row of Table 2 specifies the benchmark values used in the exercise. Production effort is set to be the contribution of capital to total output for the period of the sample. R&D intensity, growth and variance are averaged values for the period of 1980–2007. ν is calibrated using the parameter constraints of the proposition 4 and it is setting to $\nu = \gamma^2$.

Table 2: Average Moments of the Model

	e_1^{ss}	e_2^{ss}	$E\left(g^A\right)$	var^A
Data	34%	3.1%	1.83%	1.45%

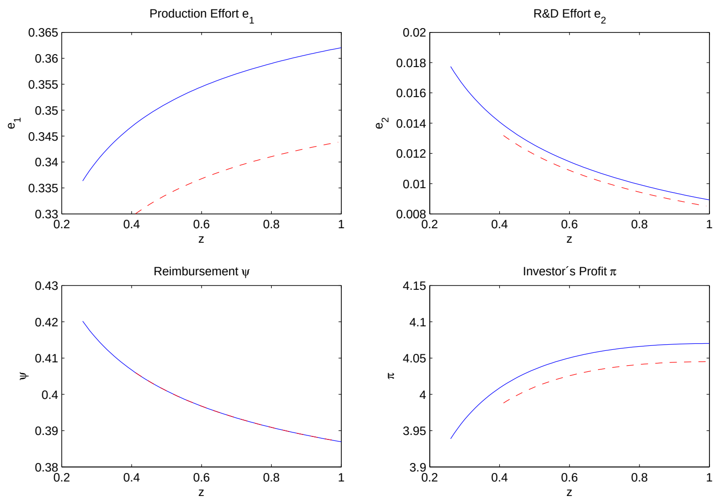
The following subsection studies the impact of the optimal contract when there is a negative idiosyncratic shock on productivity (ν) and when there is a positive shock on the substitutability parameter (γ).

4.1.1 Productivity Shocks

Now, consider a firm that receives a negative productivity shockν. In Figure 4 the dashed blue line represents the benchmark and the dotted red line shows the shock. The impact of a 5% fall in the firm's productivity leads to an average reduction by 5.06% in production effort; furthermore there is a clear effect on R&D intensity, which is reduced by an average of 4.35%. The reimbursement rate decreases slightly by around 1% while the profit curve is shifted to the right and is reduced by 0.63%.

A negative productivity shock leads to a reduction in the optimal levels of effort given to both production and R&D. From Proposition 7, it is possible to see that idiosyncratic changes in productivity proportionally affect policy functions e_1, e_2. The effect on reimbursement is cancelled out since it is the remainder of the effort allocated to production and R&D. The optimal reimbursement follows a similar shape to the non-shock case but in the range $(\hat{z}^{shock}, \bar{z}]$. Consequently, the investor's optimal level of profit is reduced and as $\hat{z}$ increases, the financial resources available to the entrepreneur are reduced.

Figure 4: Optimal Contract and Negative Productivity Shock $r = 0.04$;$\gamma = 0.95$; $\nu = 0.9$; $\nu^{shock} = 0.85$



4.1.2 Substitution Effect between Standard Production Task and R&D.

The impact of shocks on the substitutability parameter γ is shown in Figure 5 and Figure 6. When activities are less independent (i.e., rises by 5 %), they are more correlated (see Figure 5). R&D intensity increases by 4.68% and generates a spillover effect on production, which increases to 4.54%. As R&D intensity rises, the borrowing constraint is binding and fewer resources are devoted to financing entrepreneur's projects. The optimal repayment decreases as γ increases and the best strategy for the investor to reduce repayments to increase the correlation between R&D and production. This shock provides more information on the provision of effort through y. Information asymmetry is reduced and entrepreneur's output increases. The optimal contract has powerful incentive for the entrepreneur to put effort into standard production and leads to an increase in the investor's profits.

Figure 5: Effects of a Shock of Substitutability between Activities $r = 0.04$;$\gamma = 0.95$; $\nu = 0.90$; $\gamma^{shock} = 0.99$

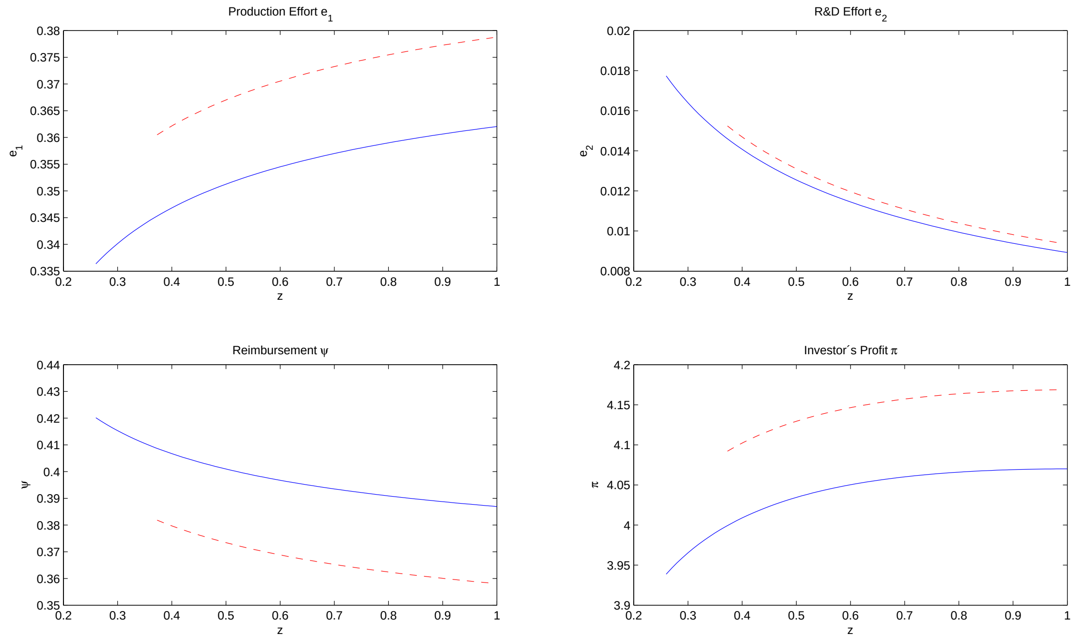
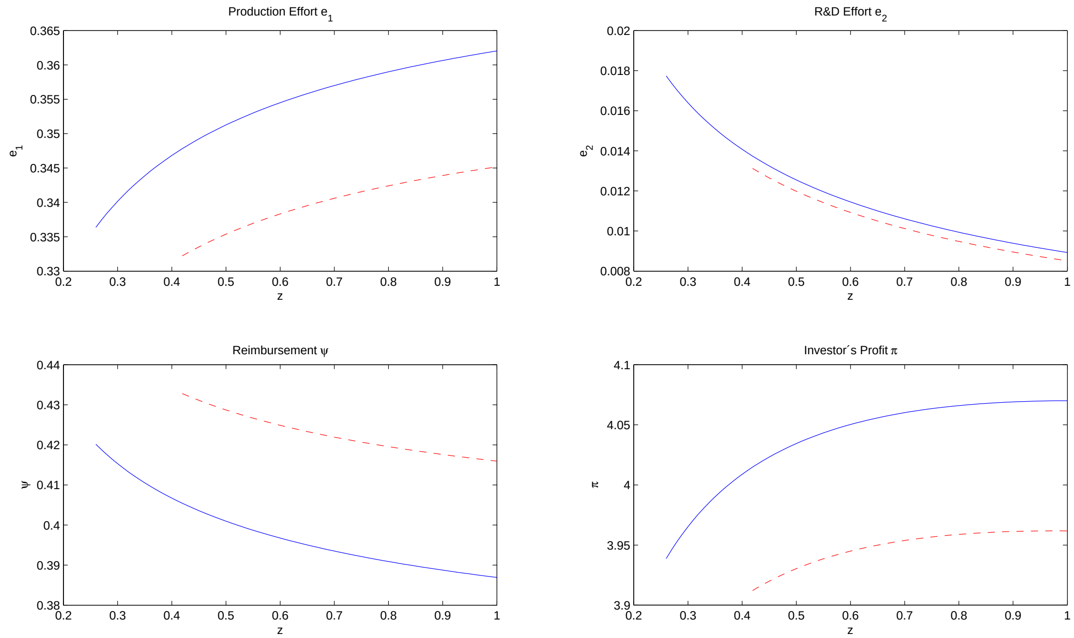


Figure 6 shows the impact when there is high substitutability $\left(\gamma^{shock} = 0.99\right)$ between standard production and R&D. In this case the effect is reversed; lower R&D intensity means less production effort and final output drops. In Figure 6, effort given to standard production falls to 4.6 % while R&D intensity is reduced to 4.54%. In this case, the optimal repayment must incentivise production. In fact, in the example, the repayment increases to reach 7.20% and profits are reduced by 2.6%. In this case, the investor left to finances 21.3% of total projects, while in the case of high substitutability, they finance 83% of projects

Figure 6: Effects of a Shock of Substitutability between Activities $r = 0.04; \gamma = 0.95$; $\nu = 0.90; \gamma^{shock} = 0.90$.



4.1.3 Combined Shock: Negative Productivity Shock ν and Less Substitutability between Tasks γ.

Here, the impact of a contraction in productivity (a reduction in of 5%) is evaluated, but at the same time projects are highly correlated (increases by 5%). In this case, R&D generates a spillover effect that outweighed the negative effect of the contraction. Nevertheless, the entrepreneur's repayments are reduced because of lower production in the standard task and the increase in R&D.

The overall impact is that effort declines by around 0.73%. R&D also falls by 0.28% while the repayments to the investor required to retain their interest in the project are reduced to around 7.04%. The investor's profit increases as the positive impact on total output is higher than the fall in the reimbursement rate.

Figure 7: Combined Shock: Negative productivity shock and Less Substitutability between Tasks

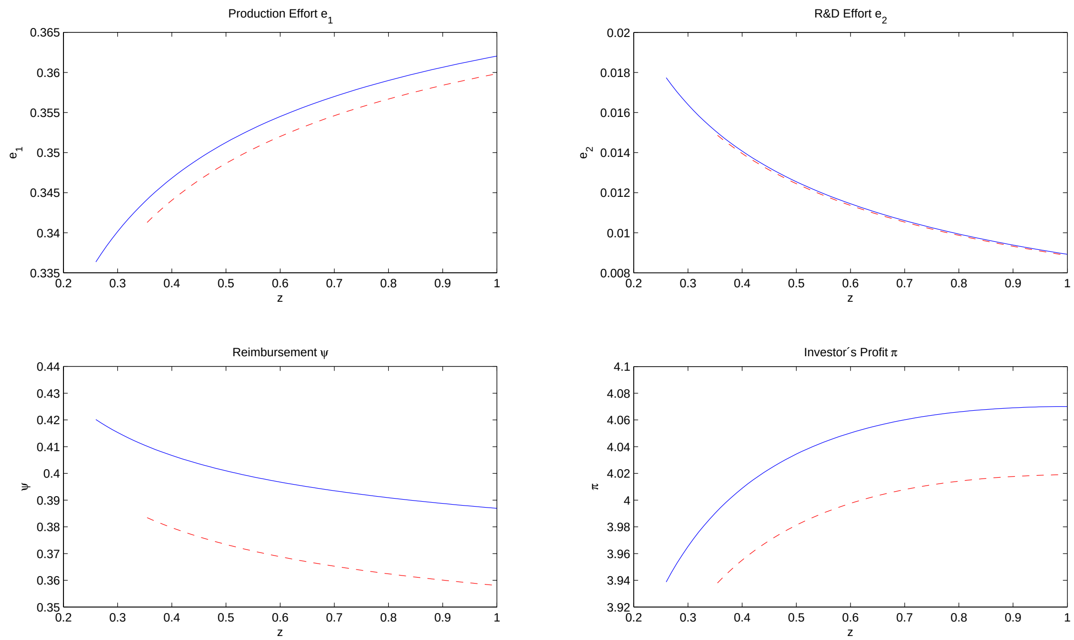


Table 3 summarises the impact on the optimal contract of the simulations presented above. The impact of the productivity shock is greater than changes in the substitutability parameter γ. A similar effect is seen in R&D intensity. The ratio e_1/e_2 is a proxy that measures the reallocation of effort between standard production and R&D. The impact of changes in ν and γ are of a similar magnitude. Reimbursements and profits are highly sensitive to changes in γ, while changes in productivity shock ν have little effect.

With respect to the firm's statistical moments, γ has a persistent impact on growth rate and variance. The impact on variance is greater for the combined shock. The last row of Table 3 represents a proxy measure of borrowing constraints. This proxy is the percentage of projects that are funded by the investor (FP). When there is high correlation between standard production and R&D, borrowing constraints are relaxed.

Table 3: Average response (av) to changes in ν and γ

	$\triangle\%\nu = -5\%$	$\triangle\%\gamma = 5\%$	$\triangle\%\gamma = -5\%$	$\triangle\%\nu = -5\%, \triangle\%\gamma = 5\%$
$\triangle\%e_1^{av}$	-5.06%	4.54%	-4.59%	-0.73%
$\triangle\%e_2^{av}$	-4.35%	4.68%	-4.54%	-0.28%
$\frac{e_2^{av}}{e_1^{av}}$	0.75%	0.05%	0.05%	-0.13%
$\triangle\%\psi^{av}$	0.07%	-7.15%	7.20%	-7.04%
$\triangle\%\pi^{av}$	-0.63%	2.39%	-2.62%	1.33%
$\triangle\%E\,(g)$	-4.79%	5.16%	-5.02%	-0.32%
$\triangle\%var$	-4.03%	4.33%	-4.20%	17.1%
$\%\,Financed\,Projects$	66.7%	82.7%	78.7%	74%

5 Concluding Remarks

A model of R&D intensity is set up in the presence of borrowing constraints. Borrowing constraints are due to the entrepreneur's misallocation of effort to different activities. R&D investment is not pledgeable for the investor, given the low value of collateral. Nevertheless, it is beneficial for firm expansion. When the entrepreneur increases the level of R&D investment, the borrowing constraint becomes binding and, therefore, some projects are not financed. Throughout this paper, it has bee showed that, at the intertemporal level, R&D resources are allocated in such a way that there are equal implicit returns between production and R&D activities.

In this model, the optimal dynamic contract between an investor and an innovative entrepreneur implies that there are incentives for both production and R&D effort. Specifically, it has been found that R&D intensity decreases as borrowing constraints are relaxed. This finding is consistent with the empirical literature and the magnitude of the impact depends on the degree of correlation between the firm's activities.

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6 Appendix: Proofs

Proof Lemma 1

For a given $s < t$ $E\left(\pi_t^E \mid \mathcal{F}_s\right) = \pi_s^E$. Setting $s > t$ and $\phi(x) = [((1-\psi_x) y_x - nc(\hat{e}_{1,x}, \hat{e}_{2,x}))]$ computing the expected value for π_t^E process, it is then:

$$\pi_t^E = r\int_0^t e^{-rx}\phi(x)\,dx + e^{-rt}E\left[r\int_t^{\tau} e^{-r(x-t)}\phi(x)\,dx + \exp(-r(\tau-t))R \mid \mathcal{F}_t\right]$$

$$\pi_s^E = r\int_0^s e^{-rx}\phi(x)\,dx + e^{-rs}E\left[r\int_s^{\tau} e^{-r(x-s)}\phi(x)\,dx + \exp(-r(\tau-t))R \mid \mathcal{F}_s\right]$$

Computing $E_t(\phi_s)$

$$E\left(\pi_s^E \mid \mathcal{F}_t\right) = E\left(r\int_0^s e^{-rx}\phi(x)\,dx \mid \mathcal{F}_t\right) + E\left[r\int_s^{\tau} e^{-rx}\phi(x)\,dx + \exp(-r(\tau-t))R \mid \mathcal{F}_t\right]$$

Solving $E_t(\phi_s - \phi_t)$

$$E\left(\pi_s^E - \pi_t^E \mid \mathcal{F}_t\right) = E\left(r\int_0^s e^{-rx}\phi(x)\,dx \mid \mathcal{F}_t\right) + E\left[r\int_s^{\tau} e^{-rx}\phi(x)\,dx \mid \mathcal{F}_t\right] - r\int_0^t e^{-rx}\phi(x)\,dx - E\left[r\int_t^{\tau} e^{-rx}\phi(x)\,dx \mid \mathcal{F}_t\right]$$

Computing the integral difference:

$$E\left(\pi_s^E - \pi_t^E \mid \mathcal{F}_t\right) = E\left(r\int_t^s e^{-rx}\phi(x)\,dx \mid \mathcal{F}_t\right) + E\left[r\int_s^{\tau} e^{-rx}\phi(x)\,dx \mid \mathcal{F}_t\right] - E\left[r\int_t^{\tau} e^{-rx}\phi(x)\,dx \mid \mathcal{F}_t\right]$$

$$E\left(\pi_s^E - \pi_t^E \mid \mathcal{F}_t\right) = E\left(r\int_t^s e^{-rx}\phi(x)\,dx \mid \mathcal{F}_t\right) + E\left[r\int_s^{\tau} e^{-rx}\phi(x)\,dx \mid \mathcal{F}_t\right] - E\left[r\int_t^{s} e^{-rx}\phi(x)\,dx \mid \mathcal{F}_t\right] - E\left[r\int_s^{\tau} e^{-rx}\phi(x)\,dx \mid \mathcal{F}_t\right]$$

Which gives: $E\left(\pi_s^E - \pi_t^E \mid \mathcal{F}_t\right) = 0$ Q.E.D.

Proof Proposition 1

It was possible to guess that the solution of the value function [6] is linear for a number of innovations n. Considering the case of interior solutions, and replacing the guess $V = f(e_1^*, e_2^*)\,n$, the Bellman equation is given by:

$$rf\left(e_1^*, e_2^*\right) = r\left[\nu e_{1,t}^* - c\left(e_1^*, e_2^*\right)\right] + f\left(e_1^*, e_2^*\right)\left[e_2\eta - \mu\right]$$

solving for $f\left(e_1^*, e_2^*\right)$

$$f\left(e_1^*, e_2^*\right) = \frac{r\left[\nu e_{1,t}^* - c\left(e_1^*, e_2^*\right)\right]}{\left[r - \left(e_2^*\eta - \mu\right)\right]}$$

Using the first order conditions results in the following system of equations:

$$[e_1] \quad \nu = c_{e_{1,t}}\left(e_{1,t}, e_{2t}\right) \tag{22}$$

$$[e_2] \quad \eta = c_{e_{2,t}}\left(e_{1,t}, e_{2t}\right) \tag{23}$$

Where policy functions are independent of firm size. Q.E.D.

Proof Proposition 2

This proof follows Sannikov (2008). As Lemma 1 proves, π^E is a martingale; from the Martingale Representation theorem for a Poisson process (see Bjork (2011) page 38). there is a predictable process h such that:

$$\pi_t^E = \pi_0^E + \int_0^t \exp\left(-rs\right) h_s W_s dN_s \tag{24}$$

where dN_s is defined in [11]. Differentiating [3] and [24] with respect to time, the following is obtained:

$$\frac{d\pi_t^E}{dt} = \exp\left(-rt\right) W_t h_t \left(\frac{dp}{dt} - e_{2,t} + \mu\right)$$

and for [24]:

$$\frac{d\pi_t^E}{dt} = r\exp\left(-rt\right)\left[\left(1-\psi_t\right) y_t - nc\left(e_{1,t}, e_{2,t}\right)\right] - \exp\left(-rt\right) W_t + \exp\left(-rt\right)\frac{dW_t}{dt}$$

Equalizing both expressions and solving for dW_t the following is obtained:

$$dW_t = r\left(W_t\left(1 - h_t\left(e_{2,t} - \mu\right)\right) - \left(1-\psi_t\right) y_t + nc\left(e_{1,t}, e_{2,t}\right)\right) dt + rW_t h_t dp.$$

Q.E.D.

Proof Proposition 3

If e_i for $i = 1, 2$ are incentives compatible during the whole path from t to ∞, then:

$$\frac{h}{r}e_2W_t - nc\left(e_1, e_2\right) \geq \frac{h}{r}\tilde{e}_2W_t - nc\left(e_1, \tilde{e_2}\right) \tag{25}$$

and

$$\left(1-\psi\right)\nu e_1 - c\left(e_1, e_2\right) \geq \left(1-\psi\right)\nu\tilde{e}_1 - c\left(\tilde{e}_1, e_2\right) \tag{26}$$

The proof for the incentive compatibility [25] is the staring point. Let fix e_1 in the truthfull path. Potential deviations of R&D effort e_2 are first analised at the intertemporal level.

$$\int_0^{\tau} exp(-rs)n_t\left[\left(\left(1-\psi_t\right)\nu e_1 - c\left(e_1, e_2\right)\right)ds\right] \geq \int_0^{\tau} exp(-rs)n_t\left[\left(\left(1-\psi_t\right)\nu e_1 - c\left(e_1, \tilde{e}_2\right)\right)ds\right]$$

Which is equivalent to

$$\int_0^{t} exp(-rs)n_t\left[\left(\left(1-\psi_t\right)\nu e_1 - c\left(e_1, e_2\right)\right)ds\right] + W_t \geq \int_0^{t} exp(-rs)n_t\left[\left(\left(1-\psi_t\right)\nu e_1 - c\left(e_1, \tilde{e}_2\right)\right)ds\right] + \widetilde{W}_t$$

Using the lemma 1, W_t and $\widetilde{W}_t$ can be expressed as martingales, then:

$$\int_0^{t} exp(-rs)n_t\left[\left(c\left(e_1, \widetilde{e}_2\right) - c\left(e_1, e_2\right)\right)ds\right] + exp(-rt)\left(W_0 + \int_t^{\tau} h_sW_sdN_s\right) \geq exp(-rt)\left(W_0 + \int_t^{\tau} h_s\widetilde{W}_sd\widetilde{N}_s\right)$$

$\int_t^{\tau} h_sW_sdN_s$ can be descomposed as: $\int_t^{\tau} h_sW_sdN_s = \int_0^{\tau} h_sW_sdN_s - \int_0^{t} h_sW_sdN_s$, therefore:

$$\int_0^{t} exp(-rs)n_t\left[\left(c\left(e_1, \widetilde{e}_2\right) - c\left(e_1, e_2\right)\right)ds\right] + exp(-rt)\left(W_0 + \int_0^{\tau} h_sW_sdN_s - \int_0^{t} h_sW_sdN_s\right)$$

$$\geq exp(-rt)\left(W_0 + \int_0^{\tau} h_s\widetilde{W}_sdN_s - \int_0^{t} h_s\widetilde{W}_sd\widetilde{N}_s\right)$$

The deviation can be expressed as :

$$dN_s = d\tilde{N}_s + \tilde{e}_{2s}ds - e_{2s}ds$$

Therefore, simplifying terms yields:

$$\int_0^t exp(-rs)n_t\left[\left(c\left(e_1,\widetilde{e}_2\right)-c\left(e_1,e_2\right)\right)ds\right]+exp(-rt)\left(\int_0^t h_sW_se_{2s}ds\right)$$

$$\geq exp(-rt)\left(\int_0^\tau h_sW_s\left(e_{2s}-\tilde{e}_{2s}\right)ds+\int_0^t h_sW_s\tilde{e}_{2s}ds\right)$$

Rearranging terms, it is obtained:

$$\int_0^t exp(-rs)\left[\frac{h_s}{r}W_se_{2s}-n_sc\left(e_{1s},e_{2s}\right)ds\right]\geq exp(-rt)\left[\int_0^\tau \frac{h_s}{r}W_s\widetilde{e}_{2s}-n_sc\left(e_{1s},e_{2s}\right)ds\right]+exp(-rt)\left(\int_0^\tau h_sW_s\left(e_{2s}-\tilde{e}_{2s}\right)ds\right)$$

The incentive compatibility constraint implies that in each period: $e_{2s}=\tilde{e}_{2s}$, therefore:

$$\frac{h_s}{r}W_se_{2s}-n_sc\left(e_{1s},e_{2s}\right)\geq\frac{h_s}{r}W_s\widetilde{e}_{2s}-n_sc\left(e_{1s},\tilde{e}_{2s}\right)$$

for each s.

Now, the converse is shown if:

$$\frac{h_s}{r}W_se_{2s}-n_sc\left(e_{1s},e_{2s}\right)\geq\frac{h_s}{r}W_s\widetilde{e}_{2s}-n_sc\left(e_{1s},\tilde{e}_{2s}\right)$$

then e_{1s},e_{2s} are incentives compatible for each s. The first step is to analyse potential deviations in effort at the intertemporal level. Let $\tilde{\pi}$ the lifetime profits of the entrepreneur that follows a strategy $\tilde{e_i}$ for $i=1,2$ up to time t, and who then switches to strategy e_i for $i=1,2$ from t to ∞. The deviation in R&D effort is then focused on. Consider the case when the agent exerts effort from $[0,t]$ by $(e_1,\tilde{e_2})$ for each t. Then, from $[t,+\infty]$ they switch to strategy (e_1,e_2). The value function of this potential deviation is given by $\tilde{\pi}^{R\&D}$ which is equivalent to:

$$\tilde{\pi}^{R\&D}=r\int_0^t exp(-rs)n_t\left[\left(c\left(e_1,e_2\right)-c\left(e_1,\tilde{e}_2\right)\right)ds\right]+\pi_t \tag{27}$$

Since [27] is a martingale, inserting equation [24] in [27] :

$$\tilde{\pi}^{R\&D}=r\int_0^t exp(-rs)n_s\left[\left(c\left(e_1,e_2\right)-c\left(e_1,\tilde{e}_2\right)\right)ds\right]+\pi_0+r\int_0^t exp(-rs)h_sW_sdN_s$$

As before, $dN_s=d\tilde{N}_s+\tilde{e}_{2s}ds-e_{2s}ds$ can be re-expressed, then:

$$\tilde{\pi}^{R\&D} = r\int_0^t exp(-rs)n_t\left[\left(c\left(e_{1s},e_{2s}\right)-c\left(e_{1s},\tilde{e}_{2s}\right)+\frac{h_sW_s}{r}\left(\tilde{e}_{2s}-e_{2s}\right)\right)ds\right]+\pi_0+r\int_0^t exp(-rs)h_sW_sd\widetilde{N}_s$$

Derivating with respect to time:

$$\frac{d\tilde{\pi}^{R\&D}}{dt} = rexp(-rt)n_t\left[\left(c\left(e_{1t},e_{2t}\right)-c\left(e_{1t},\tilde{e}_{2t}\right)-\frac{h_tW_t}{r}\left(e_{2t}-\tilde{e}_{2t}\right)\right)dt\right]+exp(-rt)h_tW_t\left(\frac{dp}{dt}-\tilde{e}_{2t}+\mu\right)$$

Under the incentive compatible strategy, the drift of the process $\tilde{\pi}^{R\&D}$ is negative.

Therefore, if the pairs of effort (e_1, e_2) are incentive compatible, then the drift of the process $\tilde{\pi}^{R\&D}$ is negative. In fact $\tilde{\pi}^{R\&D}$ under the process $\{e_1, \tilde{e}_2\}$ satisfies:

$$E_{e_1,\tilde{e_2}}\left(\tilde{\pi}^{R\&D}\right) \leq E_{e_1,e_2}\left(\pi\right)$$

which implies strategy $e_2 \succ \tilde{e}_2$.

Now, consider the case of change in production effort, from e_1 to $\tilde{e_1}$. If e_1, e_2 are incentive compatible in the interval $[0, \tau]$, then:

$$\int_0^\tau exp(-rs)n_t\left[\left(\left(1-\psi_t\right)\nu e_1 - c\left(e_1,e_2\right)\right)ds\right] \geq \int_0^\tau exp(-rs)n_t\left[\left(\left(1-\psi_t\right)\nu \tilde{e}_1 - c\left(\tilde{e}_1,e_2\right)\right)ds\right]$$

Re-arranging terms:

$$\int_0^\tau exp(-rs)n_t\left[\left(\left(1-\psi_t\right)\nu\left(e_1-\tilde{e}_1\right)-\left[c\left(e_1,e_2\right)-c\left(\tilde{e}_1,e_2\right)\right]\right)ds\right] \geq 0$$

For each t we have:

$$\left(1-\psi\right)\nu e_1 - c\left(e_1,e_2\right) \geq \left(1-\psi\right)\nu\tilde{e}_1 - c\left(\tilde{e}_1,e_2\right)$$

The next step is to analyse whether the agent follows an strategy in which the agent exerts effort from $[0, t]$ by $(\tilde{e}_1, e_2)$ for each t. Then, from $[t, +\infty]$ they switch to strategy (e_1, e_2). The value function of this potential deviation is given by $\tilde{\pi}^{prod}$ which is equivalent to:

$$\int_0^\tau exp(-rs)n_t\left[\left(\left(1-\psi_t\right)\nu\tilde{e}_1 - c\left(\tilde{e}_1,e_2\right)\right)ds\right]+\pi_t^E-\int_0^\tau exp(-rs)n_t\left[\left(\left(1-\psi_t\right)\nu e_1 - c\left(e_1,e_2\right)\right)ds\right]$$

Using the fact that π_t^E is a martingale then:

$$\pi^{prod} = -\int_0^{\tau} exp(-rs)n_t\left[\left((1-\psi_t)\,\nu\,(e_1-\tilde{e}_1)+\left[c\,(e_1,e_2)-c\,(\tilde{e}_1,e_2)\right]\right)ds\right]+\pi_0^E-\int_0^{\tau} exp(-rs)dN_s$$

Then if the incentive compatibility condition [26], it is then obtained:

$$E_{\tilde{e}_1,e_2}\left(\pi^{prod}\right) \leq E_{e_1,e_2}\left(\pi\right)$$

Notice that the global constraints are also satisfied because the drift in both cases is positive, $E_{e_1,e_2}(\pi) \geq 0$. Q.E.D.

Proof Proposition 4

The first case to be analised is that when there is not R&D effort. Then, the Bellman equation is as follows:

$$r\pi^I(z) = \max_{e_1,\psi} r\left(\psi e_1 \nu\right) + r\pi_z^I\left[z-(1-\psi)\,e_1\nu + c\,(e_1)\right]$$

Then, the first order conditions are:

$$[e_1]: \quad \nu\psi + \pi_z^I\left(-(1-\psi)\,\nu + e_1\right) = 0 \tag{28}$$

$$[\psi]: \quad \pi_z^I = -1 \tag{29}$$

As the contract is incentive compatible for each implementation of e_1, then $e_1^{prod} = (1-\psi)\,\nu$, inserting the first order condition [28] we obtain $\psi = 0$; therefore, the investor's profits are negative when the agent only exerts effort in production. Consequently, the optimal level of production effort is constant and equal to the productivity parameter ν, $e_1^{prod} = \nu$, which is not profitable for the investor.

When the entrepreneur exerts effort in both activities, the Bellman equation is given by [17]. The first- order conditions are:

$$[e_1]: \quad \nu\psi + \pi_z^I\left((1-\psi)\,\nu + e_1 + \gamma e_2 - ze_2 r\gamma\right) + e_2\pi_z^I\left(\tilde{z}\right) zr\gamma = 0 \tag{30}$$

$$[e_2]: \quad -e_2\left(1-2zr\right) - e_1\gamma\left(1-r\right) + \frac{\pi\left(\tilde{z}\right)-\pi\left(z\right)}{r} + e_2\pi_z\left(\tilde{z}\right) zr = 0 \tag{31}$$

$$[\psi]: \quad \pi_z^I = -1 \tag{32}$$

It is assumed that the expected profit with technological jumps is given by $\mathrm{E}(\pi(\tilde{z})) = \pi(z)(1+rh)$, hence, inserting the first order condition [32] in the equations [28] and [31], yield the following system of equations:

$$\frac{\nu}{r} - \frac{e_1}{r} - \frac{\gamma}{r}e_2 + ze_2\gamma + ze_2\gamma + r^2\gamma e_2 z + r^2\gamma^2 e_1 z - \gamma z = 0 \tag{33}$$

and

$$-e_2(1-2zr) - e_1\gamma(1-r) - zr(e_2+\gamma e_1) + \left(r^2 e_2^2 + e_2\left(r^2 e_1\gamma - 1\right)\right)zr = 0 \tag{34}$$

Using [34] it is solved for e_1:

$$\frac{\nu - r\gamma z + e_2\left(-\gamma + z\gamma r + r^3\gamma z\right)}{1 - r^3\gamma^2 z} = e_1 \tag{35}$$

Define $D = \nu - r\gamma z$, $F = -\gamma + z\gamma r + r^3\gamma z$ and $G = 1 - r^3\gamma^2 z$.
Rewrite [35] as

$$\frac{D + e_2(F)}{G} = e_1 \tag{36}$$

Replacing in [34] and simplifying, the expression for e_2 is:

$$e_2^{sb} = \left(\frac{\sqrt{A+B} + D}{2C}\right) \tag{37}$$

$A = \left(\gamma\left(r^2 z\left(\gamma\left(r^2+2\right)+\nu r\right) - 2\gamma r - \gamma r^2 z^2\left(r(\gamma+r)+1\right)+\gamma\right) + r - 1\right)^2$
$B = -4\left(r^3 z\left(\gamma^2 rz - \gamma^2 + 1\right)\right)\left(\gamma\left(2r - 1 - rz\right)\left(\nu - \gamma^2 z\right)\right)$
$D = -\gamma^2\left(r^2 z(z-2) + 2r + r^4(z-1)z - 1\right) - \gamma - \nu\gamma r^3 z^2 + 1$
$C = \left(r^3 z\left(\gamma^2 rz\gamma^2 + 1\right)\right)$

e_2^{sb} is convex, since $C > 0$. To determine the sign of e_2^{sb}, the sign of each component of the numerator and denominator is analysed as follows:

sign $A > 0$, *sign* $B > 0$ that is given by the following parameter restrictions:

$$4\left(r^3 z\left(\gamma^2 rz - \gamma^2 + 1\right)\right) > 0 \tag{38}$$

$$\gamma\left(-r(z-2) - 1\right) < 0 \tag{39}$$

$$\left(\nu - \gamma^2 z\right) > 0 \tag{40}$$

$$-4\left(r^3 z\left(\gamma^2 r z-\gamma^2+1\right)\right)\left(\gamma\left(-r\left(z-2\right)-1\right)\right)\left(\nu-\gamma^2 z\right)>0 \tag{41}$$

Finally, $sign\ D>0$ according to the next conditions:

$$\gamma^2\left(r^2 z\left(k+z-2\right)\right)<0 \tag{42}$$

$$2r+r^4\left(z-1\right)z-1<0 \tag{43}$$

$$-\nu\gamma r^3 z^2<0 \tag{44}$$

$$-\gamma^2\left(r^2 z\left(z-2\right)+2r+r^4\left(z-1\right)z-1\right)+1>1 \tag{45}$$

$$0<|-\gamma-\nu\gamma r^3 z^2|<1$$

Then $e_2^{sb}>0$.

Next step is to obtain the derivative with respect to z $e_{2,z}$.

$$e_{2,z}=\left[\frac{\partial\left[\sqrt{A+B}+D\right]}{\partial z}2C-\frac{\partial 2C}{\partial z}\left[\sqrt{A+B}+D\right]\right]/4C^2$$

$\frac{\partial\left[\sqrt{A+B}+D\right]}{\partial z}=\frac{1}{2}(A+B)^{-1/2}\left(A_z+B_z\right)+D_z$

Start with A_z :

$$A_z=2r^2\gamma(r\nu+(2+r^2)\gamma-2z\gamma(1+r(r+\gamma)))\{-1+r+\gamma[\gamma-2r\gamma+r^2 z(r\nu+(2+r^2)\gamma)-r^2z^2\gamma(1+r(r+\gamma))]\}<0$$

$$B_z=-4\gamma r^3(\nu(\gamma^2+\gamma^2r^2(4-3z)z-2r(\gamma^2+z-1)-1)+\gamma^2 z(-2\gamma^2+2\gamma^2r^2z(2z-3)+r(4\gamma^2+3z-4)+2))\Rightarrow 0$$

$$D_z=\gamma(-r^2)(\gamma r^2(2z-1)+2r\nu z+2\gamma(z-1))>0$$

For the purpose of simplicity, the parameters γ, r, ν have been set to the values of steady-state and varying z, then $e_{2,z}<0$.

Let e_2^{sb} be the optimal value for the effort in R&D. The effort in production e_1 is:

$$e_1^{sb} = \frac{\gamma\left(e_2^{*}\left(r^3 z + rz - 1\right) - \gamma z\right) + \nu}{1 - \gamma^2 r^3 z} \tag{46}$$

Given the following conditions proved above $e_2^{sb} > 0$, $e_{2,z}^{sb} < 0$ y $e_{2,zz}^{sb} > 0$, $e_{1,z}^{sb}$ is computed as:

$$e_{1,z}^{sb} = \frac{\gamma\left(-\left(r^3 z + rz - 1\right) e_{2,z}^{sb}\left(r^3\gamma^2 z - 1\right) + e_2^{sb}\left(r - r^3\left(\gamma^2 - 1\right)\right) + \gamma\left(\nu r^3 - 1\right)\right)}{\left(r^3\gamma^2 z - 1\right)^2} \tag{47}$$

Restating e_{1z} considering that the minimum value that ν can take $\nu = \gamma^2$ then:

$$e_{1,z}^{sb} = \frac{\gamma\left(\nu r^3 - 1\right)\left[-\left(r^3 z + rz - 1\right) e_{2,z}^{sb} + 1\right] + e_2^{sb}\left(r - r^3\left(\gamma^2 - 1\right)\right)}{\left(r^3\gamma^2 z - 1\right)^2}$$

Since the following conditions are satisfied:

$$e_2^{sb}\left(r - r^3\left(\gamma^2 - 1\right)\right) > 0$$

$$-\left(r^3 z + rz - 1\right) e_{2,z}^{sb} + 1 < 0$$

$$\gamma\left(\nu r^3 - 1\right) < 0$$

$$\gamma\left(\nu r^3 - 1\right)\left[-\left(r^3 z + rz - 1\right) e_{2,z}^{sb} + 1\right] > 0$$

$$\gamma\left(\nu r^3 - 1\right)\left[-\left(r^3 z + rz - 1\right) e_{2,z}^{sb} + 1\right] + e_2^{sb}\left(r - r^3\left(\gamma^2 - 1\right)\right) > 0$$

Thus $e_{1,z}^{sb} > 0$.

The second derivative of e_1 with respect to z is

$$e_{1,zz}^{sb} = \frac{1}{\left(r^3\gamma^2 z - 1\right)^3}\left(-\gamma\left(r^3 z + rz - 1\right) e_{2,z,z}^{sb}\left(r^3\gamma^2 z - 1\right)^2 + 2r\gamma\left(r^2\left(\gamma^2 - 1\right)\text{-}1\right)\left(r^3\gamma^2\left(e_2^{sb} - z e_{2,z}^{sb}\right) + e_{2,z}^{sb}\right) - 2r^3\gamma^4\left(\nu r^3 - 1\right)\right.$$

According to the parameter restrictions, it is then as follows:

$$-\gamma\left(r^3 z + rz - 1\right) e_{2,z,z}^{sb}\left(r^3\gamma^2 z - 1\right)^2 > 0$$

$$2r\gamma\left(r^2\left(\gamma^2-1\right)\text{-}1\right)>0$$

$$\left(r^3\gamma^2\left(e_2^{sb}-ze_{2,z}^{sb}\right)+e_{2,z}^{sb}\right)-2r^3\gamma^4\left(\nu r^3-1\right)>0$$

$$-\gamma\left(r^3z+rz-1\right)e_{2,z,z}^{sb}\left(r^3\gamma^2z-1\right)^2+2r\gamma\left(r^2\left(\gamma^2-1\right)\text{-}1\right)\left(r^3\gamma^2\left(e_2^{sb}-ze_{2,z}^{sb}\right)+e_{2,z}^{sb}\right)-2r^3\gamma^4\left(\nu r^3-1\right)>0$$

$$\frac{1}{\left(r^3\gamma^2z-1\right)^3}<0$$

Therefore $e_{1,zz}^{sb}<0$ and the result yields Q.E.D.

Proof Proposition 5

Suppose that $\pi^I(z)$ is a differentiable continuous function. Consider $z^*<z$ such that $z^*\in[\hat{z},\bar{z}]$; from the optimal contract it is known that π_z^I is non-increasing, meaning that there is $z_1\in[z^*,\bar{z}]$ such that $\pi_{z_1}^I(z_1)\leq\pi_z^I(z)$. Integrating both sides of the inequality, it is then:

$$\int_{z^*}^{z}\left(\pi_{z_1}^I(z_1)\right)dz_1\leq\int_{z^*}^{z}\left(\pi_z^I(z)\right)dz_1$$

Using the fundamental theorem of calculus:

$$\pi^I(z)-\pi^I(z^*)\leq\pi^I(z^*)(z-z^*) \tag{48}$$

Re-arranging the terms the definition of concavity is given around z^*, $\pi^I(z)\leq\pi^I(z^*)(z-z^*)+\pi^I(z^*)$. Q.E.D.

Proof Proposition 6

Consider the aggregate growth rate as $\tilde{g}=\int_{\hat{z}}^{\bar{z}}g(z)\,dz$ which is exponentially distributed with rate $e_2=\int_{\hat{z}}^{\bar{z}}e_2(z)\,dz$. The first step is to compute the survival function. For any $t>0$ so:

$$P(M>t;g,z)=\int_0^t\left(\exp\left(e_2(z)-\mu\right)x\right)\left(e_2(z)-\mu\right)dx$$

Which is equivalent to:

$$P\left(M>t;g,z\right)=1-\left(\exp\left(e_{2}\left(z\right)-\mu\right)t\right)$$

The survival function is defined as:

$$S\left(g,z\right)=1-P\left(M>t;g,z\right)=\left(\exp\left(e_{2}\left(z\right)-\mu\right)t\right)$$

The survival function collapses the distribution of the firm's growth over time. Therefore the expected growth of a firm that survives is given by:

$$g^{s}\left(z\right)=\left(\exp\left(e_{2}\left(z\right)-\mu\right)t\right)\left(\exp\left(e_{2}\left(z\right)-\mu\right)t-1\right)$$

Then, aggregate the survival function over the range of financial resources that measure the aggregate growth of a firm that survive:

$\widetilde{g^{s}}=\int_{\hat{z}}^{\bar{z}}\left(\exp\left(e_{2}\left(z\right)-\mu\right)t\left(\exp\left(e_{2}\left(z\right)-\mu\right)t-1\right)\right)dz$. Compute $\tilde{g_{z}^{s}}$ using the Leibniz formula:

$$\tilde{g_{z}^{s}}=\int_{\hat{z}}^{\bar{z}}g_{z}^{s}\left(z\right)ds-g^{s}\left(\hat{z}\right)\hat{z}_{z}$$

Then, compute $g_{z}^{s}\left(z\right)$ as: $g_{z}^{s}\left(z\right)=g_{e_{2}}^{s}\left(z\right)e_{2,z}\left(z\right)$. The marginal response of the R&D effort on the rate of growth of a firm that survives is given by:

$$g_{e_{2}}^{s}\left(z\right)=\left(1+g\right)\left(t\left(1+2g\right)\right)>0$$

then from Proposition 4, it is known in the optimal contract $e_{2,z}\left(z\right)<0$ for $z\in\left[\hat{z},\bar{z}\right]$ then $g_{z}^{s}\left(z\right)<0$. The measure of borrowing constraint implies that $\hat{z}_{z}>0$ therefore $\tilde{g_{z}^{s}}<0$.

In order to analyze the variance, focus on the aggregate variance for all realizations of z $v\tilde{a}r\left(z\right)=\int_{\hat{z}}^{\bar{z}}var\left(z\right)ds$, and compute:

$$v\tilde{a}r_{z}\left(z\right)=\int_{\hat{z}}^{\bar{z}}\left[var_{z}\left(z\right)\right]ds-var\left(\hat{z}\right)\hat{z}_{z}$$

In the same way, the derivative $var_{z}\left(z\right)=var_{e_{2}}\left(z\right)e_{2,z}\left(z\right)$ was found by using the chain rule. Therefore, Lemma 2 is used to find $var_{e_{2}}\left(z\right)$:

$$var_{e_2}(z) = \frac{1+g}{e_2-\mu}\left(t\left(e_2+\mu\right) - 2\mu g\left(e_2-\mu\right)\right) > 0$$

Notice that $t\left(e_2+\mu\right) - 2\mu g\left(e_2-\mu\right) > 0$, which implies $var_{e_2}(z) > 0$. In addition, known from proposition 4 $e_{2,z}(z) < 0$ then $var_z(z) < 0$. In the case of borrowing constraint: $\hat{z}_z > 0$ and if the $var(\hat{z}) > 0$ then the result yields Q.E.D.