Testable Restrictions on the Equilibrium Manifold under Random Preferences.*

Andrés Carvajal†

Brown University and Banco de la República

October 29, 2002

Abstract

General equilibrium theory was criticized for its apparent irrefutability, as seemingly implied by the Sonnenschein-Mantel-Debreu theorem. This view was challenged by Brown and Matzkin (1996), who showed the existence of testable restrictions on the equilibrium manifold. Brown and Matzkin, however, maintain the assumption that individual preferences are invariant (against psychological evidence). I consider the Brown-Matzkin problem under random preferences: for each profile of endowments one observes a distribution of prices; does there exist a probability distribution of preferences that explains the observed distributions of prices via Walrasian equilibria? I argue that even under random utilities general equilibrium theory is falsifiable.

Keywords: General equilibrium; Revealed preferences; Random utility; Testable restrictions.

JEL classification: D12, D70, D51.

*Based on chapter five of my doctoral dissertation "On Individually-rational Choice and Equilibrium," submitted at the Department of Economics, Brown University. I would like to thank Herakles Polemarchakis for introducing me to the topic and for invaluable suggestions. Comments by Indrajit Ray, Susan Snyder, Rajiv Vohra, Moshe Buchinsky and seminar participants at Brown, Banco de la República, the 2002 Latin American Meeting of the Econometric Society and Universidad Javeriana are also gratefully acknowledged. All remaining errors are mine. Brown University and Banco de la República provided financial support.

†Address: Cra 7 # 14-78 piso 11, Bogotá, Colombia. Tel: 571-343-0529. Fax: 571-342-1804. E-mail address: acarvaes@banrep.gov.co

1 Introduction:

Does general equilibrium theory constitute scientific knowledge? According to a prominent point of view in epistemology, commonly referred to as "falsificationism", one can only answer this question by determining whether or not the theory implies empirical regularities that can be –moreover, that are *ex ante* considered likely to be– refuted by real data. The most important defender of this theory, Karl Popper, argued that scientific discovery should be attempted through the following process: (i) the internal consistency of a theory must be formally checked, to verify that it contains no logical inconsistencies; (ii) the logical principles of the theory must be distinguished from its empirical implications; (iii) the theory must be compared with alternative existing theoretical knowledge that has not been refuted by empirical evidence, in order to ascertain whether it can explain phenomena that cannot be explained by the existing knowledge; (iv) finally, the theory must be submitted to tests of its empirical implications, in order for it to be corroborated (but not verified) or refuted. Interesting tests are those that are "harsh," in the sense that, *a priori*, the theory would appear likely to fail them. If a theory fails a test, and there exists no reasonable excuse that can itself be tested, then the theory should be abandoned.

For general equilibrium, the developments of Arrow, Debreu and McKenzie during the Fifties took care of step (i): the principles of the theory were most clearly presented and their logical consistency was highlighted by their existence results. The work of Sonnenschein, Mantel and Debreu during the Seventies, however, led many an economist to believe that the rest of the falsificationist process could not be followed for general equilibrium theory, as it was generally understood that it did not impose any (strong) empirical regularity that could be refuted with data, unless one observed individual behavior, which is unlikely.

This generalized understanding that general equilibrium theory was unfalsifiable was problematic from both a scientific point of view and an economic policy perspective, as it implied that the foundations of most theoretical developments in economics and of many economic policy recommendations lacked scientific character and could only be believed out of faith in the theory.

Such pessimistic view, however, was challenged in 1996 by Brown and Matzkin, who exploited an existing tension between the two fundamental concepts of the theory, namely individual rationality and market clearing, to show that whenever individual budgets are observed, the theory imposes nontrivial testable restrictions on the prices that can arise as Walrasian equilibria, thus showing that the theory is falsifiable even without the observation of individual choices.

The argument of Brown and Matzkin making the case for falsifiability of the theory crucially assumes that individual preferences are invariant and uses revealed-preference theory in order to argue the existence of data which is inconsistent with general equilibrium. This feature of the argument may seem natural in economics, but opens the door to strong criticism from other disciplines.

From a philosophical standpoint, Krober-Riel (1971) has written that

"...revealed-preference theory tries to leave the problematic relations between introspectively perceivable preferences and buyer behavior out of consideration: revealed-preference interprets the observable purchasing acts as 'revealed preferences', and only those revealed preferences are accepted as the starting point for the calculation of demand... It is assumed that the empirical relevance of demand theory is thereby improved... However if empirical support is required, revealed-preference theory proves to have as little foundations in reality as classical theory... The consumer in revealed-preference theory is a 'defined individual', a special kind of homo oeconomicus, whose rationality is presumed axiomatically... The alleged advantages of increased empirical relevance and especially the 'behavioristic' point of view of this theory prove to be linguistic declarations without factual meaning..."

And Hausman (2000) further argued that:

"The notion of 'revealed preference' is unclear and should be abandoned. Defenders of the theory of revealed preference have misinterpreted legitimate concerns about the testability of economics as the demand that economists eschew references to (unobservable) subjective states."

Moreover, a straightforward criticism is evident from the falsificationist process itself. What if one finds a data set to be inconsistent with the general equilibrium theory using the test offered by Brown and Matzkin? Is there a reasonable excuse to explain such failure? The excuse that first comes to mind is precisely that individual preferences are not really invariant. If that is the case, individual rationality and market clearing may still be consistent with the data. The question is, then, how reasonable is the assumption of variable preferences? Significant research in human behavior seems to have convinced psychologists that it is indeed very reasonable: in the Fifties, mathematical psychologists launched a search for a theory where human preferences are probabilistic in nature and, therefore, so are human behavior and choice.

Luce (1959) first wrote that:

"A basic presumption... is that choice is best described as a probabilistic, not an algebraic phenomenon... The probabilistic philosophy is by now a commonplace is much of psychology, but it is a comparatively new and unproven point of view in utility theory. To be sure, economists when pressed will admit that the psychologist's assumption is probably the more accurate, but they have argued that the resulting simplicity warrants an algebraic idealization."

Similarly, Block and Marshak (1960) wrote that:

"In interpreting human behavior there is a need to substitute 'stochastic consistency of choices' for 'absolute consistency of choices.' The latter is usually assumed in economic theory, but is not well supported by experience."

Accordingly, Luce and Supes (1965) justify the choice of a probabilistic understanding of human behavior adopted by psychologists by saying that:

"Historically, the algebraic theories were studied first, and they have been used in economics and statistics almost exclusively. The probabilistic ones are largely the product of psychological thought, forced upon [psychologists] by the data [they] collect in the laboratory."

Does allowing for variable preferences imply, again, that general equilibrium theory is unfalsifiable? This paper incorporates the theory of random utility to the problem of deriving testable implications of general equilibrium theory without the observation of individual choice. My results show that even under random preferences the theory imposes nontrivial restrictions on probabilistic distributions of prices, which are necessary and sufficient for them to be consistent with observed profiles of individual endowments and general equilibrium.

The paper is organized as follows: in section 2, I give a brief survey of relevant literature and distinguish the problem dealt with here from problems and results obtained elsewhere. Section 3 further motivates the problem, by introducing an example in which I illustrate both the argument of Brown and Matzkin for inconsistency of data and theory, as well as its criticism from the point of view of mathematical psychology. Section 4 then lays down the problem in the specific way in which it will be dealt with here, and introduces the assumptions that I make. In section 5, I obtain the first results, which constitute a characterization of data that are consistent with general equilibrium under random preferences, via the existence of disaggregate variables satisfying necessary and sufficient conditions for their rationalizability as consistent with general equilibrium. Given that this characterization is mediated by existential quantifiers, it fails to provide the basis for a direct test and it is unclear whether the null hypothesis of consistency can ever be refuted. Section 6, which introduces two examples of non-rationalizable data sets illustrating separate dimensions of the problem, makes the case for refutability, while section 7 provides another characterization of rationalizability and uses standard results in quantifier elimination to determine the abstract form that restrictions on the data set alone ought to have.

2 Review of literature:

The first study of the problem of falsifiability of general equilibrium theory without observation of individual choices was Sonnenschein (1973), where the following problem was posed: suppose that one observes a function mapping prices into quantities of commodities; what conditions must this function satisfy if it is to be the aggregate excess demand function of an exchange economy under standard assumptions? Well-known necessary conditions are continuity, homogeneity of degree zero and Walras' law. The surprising result was that these very mild conditions exhaust all the restrictions of the theory, as shown by Mantel (1974) and Debreu (1974): for any function that satisfies these three conditions, there exists an economy, with at least as many consumers as commodities, such that, away from zero prices, the function is its aggregate excess demand function. This result is commonly referred to as the Sonnenschein-Mantel-Debreu theorem.[1]

[1] Mas-Collel (1977) showed that there are no restrictions on the set of equilibrium prices of an economy, Diewert (1977) showed that there are some restrictions on the derivatives of the aggregate excess demand and Geanakoplos and Polemarchakis (1980) showed that these are all the restrictions. A similar result for market demand functions was shown by Diewert (1977) and Mantel (1977). Andreu proved that a conclusion similar to the Sonnenschein-Mantel-Debreu applies to finite subsets of prices. Recently, Chiappori and Ekeland (1999) showed that the Sonnenschein-Mantel-Debreu extends to the whole market demand function, under smoothness assumptions. For a recount of the earlier part of this literature, see Shafer and Sonnenschein (1982).

The conclusion was formed that if the condition that there are at least as many consumers as there are commodities is acceptable, then the restrictions of utility maximization disappear when one does not observe individual choices. This interpretation was challenged by Brown and Matzkin (1996), who showed that general equilibrium theory is falsifiable, even without observing individual choices, provided that there exists information about individual budgets. The novelty of their approach resided in that they did not analyze the aggregate excess demand function, which from an empirical point of view is inconvenient, as under the general equilibrium hypothesis it can only be observed precisely when it vanishes, but focused on the equilibrium manifold, where variations of individual endowments are accounted for. By varying individual endowments, Brown and Matzkin showed a conflict that may arise between the two principles that constitute the basis of general equilibrium: individual rationality and market clearing. Specifically, they showed an important tension between aggregate feasibility and individual-wise satisfaction of the axioms of revealed preference, the first of which is necessary condition for market clearing, and the second of which is equivalent to individual rationality. This tension implied that not every data set of individual endowments and prices can be rationalized as coming from observations of Walrasian equilibria in an exchange economy under standard assumptions.

A similar approach, where individual endowments are taken into account, was taken by Chiappori et al (2002), with the difference that they consider the whole of the equilibrium manifold, rather that just some finite subset of it. They find that "whenever data are available at the individual level, then utility maximization generates very stringent restrictions upon observed behavior, even if the observed variables are aggregate (e.g. aggregate excess demand or equilibrium prices)." Furthermore, under the extra assumption that individuals have preferences such that income effects do not vanish, they show that all the restrictions of individual rationality are preserved upon aggregation, since it is possible to recover individual preferences from the equilibrium manifold (at least locally), uniquely up to ordinal equivalence. They also show that some individual level information is necessary for falsification, since any smooth manifold can be locally rationalized as resulting from utility maximizing agents, whenever their number is at least as large as the number of commodities and redistribution of endowments is allowed.

In this paper, I take the same approach as in Brown and Matzkin (1996), which requires the observation of only a finite data set. However, I allow for random individual preferences, so that even for a given profile of endowments, equilibrium prices are random in nature. Another paper in which the problem of falsifiability of general equilibrium theory in a nondeterministic environment is studied is Kubler (2001). There, it is studied whether intertemporal general equilibrium with incomplete markets imposes restrictions on prices of commodities and assets, given a stochastic process of dividends and aggregate endowments. It is found that if one restricts individual preferences over the tree of the economy to be additively-separable, expected-utility preferences, then there do exist testable restrictions.[2]

So as to avoid confusion, I will now highlight the differences between the problem studied by Kubler and mine. Conceptually, two differences are clear. Kubler's problem is intertemporal, and agents have to make their decisions under uncertainty. The problem I study here, on the other hand, *has no intertemporal features* and although individual preferences are assumed to be random, *my agents never decide under uncertainty.* That is to say that instead of an

[2] See, nonetheless, Constantinides and Duffie (1996) and Krebs (2001), where it is argued that further idiosyncratic risk may destroy these restrictions. Other extensions of the Brown-Matzkin analysis, which are less related to the matters dealt with here are Snyder (1999) and Carvajal (2002a).

intertemporal problem with uncertainty, my agents face a set of independent problems, and at each one of these problems they make their decisions fully aware of their preferences (which have already realized), and with no consideration for other problems in the set. Besides, Kubler assumes the observation of a joint stochastic process of prices, aggregate endowments and dividends for a given event tree and a set of agents, while I, also taking as given the sets of events of nature and agents, assume a very different structure for my data set: for each one of a set of profiles of individual endowments, I assume that one observes a probability measure on the space of prices. There is no sense of sequentiality in either the set of endowments or the way in which prices are observed.

3 Motivation: revealed preference and the random utility criticism.

Suppose that one has gathered data on individual endowments and prices for a two-consumer, two-commodity exchange economy. Suppose that two profiles of endowments, $e, e' \in \left(\mathbb{R}^2_{++}\right)^2$, have occurred, and that for *both* of them, the vectors of prices $\widetilde{p}, \widehat{p} \in \mathbb{R}^2_{++}$ have been observed in the market.[3] Figure 1 illustrates the Edgeworth boxes of these endowments and show the budget lines implied by the vectors of prices. Figure 2 overlaps the previous figures.

This case embeds the example in Brown and Matzkin (1996) of a non-rationalizable data set. The argument is straightforward. No strongly concave, strictly monotone utility function can be consistent with the choices that individual 1 would have to be making, given his budgets and the aggregate feasibility constraint. To see this, consider figure 3, where the regions of the budget sets of

[3] The fact that prices have been the same for both endowments is irrelevant for the argument. It just makes the figures look simpler.

individual 1, for the endowment-price combinations $(e, \widehat{p})$ and $(e', \widetilde{p})$ which are consistent with the aggregate endowments have been highlighted. If these prices were equilibria, and hence markets cleared, the demands of individual 1 would have to be in violation of the Weak Axiom of Revealed Preference (WARP), a well-known necessary condition for individual rationality. Hence, this data set is not rationalizable in the Brown-Matzkin context.

To many a mathematical psychologist, this argument would appear unsatisfactory, as it relies heavily in the revealed preference paradigm, which gives the preferences of individuals an absolute, rigid character, something that is difficult to accept based on observations of human behavior or even on simple introspection. The assumption that preferences of an individual never change and are perfectly known by himself, which is very common in economics, is seldom thought to be realistic in psychology.

In this context, one could, tentatively, argue that there is nothing wrong with the data that was deemed nonrationalizable in the previous argument. If one just accepts that there may exist two pairs of preferences (each pair being composed of a preference relation for each individual,) and that preferences of individuals may have changed as a result of changes in nature, that one cannot observe, then there is no objection to arguing that prices $\widetilde{p}$ arise in both economies when one of the two pairs of preferences is realized, which is illustrated in the upper panel of figure 4, whereas prices $\widehat{p}$ are consistent with the other pair of preferences, as the lower panel of the figure. In both panels, demands that would be consistent with revealed-preference axioms and market clearing have been highlighted.[4]

The purpose of this paper is to show that, even under random utility assump-

[4] For illustration purposes, the budget line of individual 2 for endowments e has been copied in dashes relative to the origin of the box under e'. This shows that 2 need not violate the axioms of revealed preferences, whereas from the continuous budget lines it is clear that 1 does not either.

tions, general equilibrium theory can still be refuted. Of course, the question of testability is not interesting if whenever an anomaly in the predictions of a theory is found, the researcher allows herself or himself to explain it as an unobservable change in the environment to which the theory applies. If that were the case, any prediction within the algebraic scope of the theory would be rationalizable, and the testability question would be meaningless. One must then impose some stability to the environment of the theory. Here, this stability will be given by assuming that we observe, for each profile of endowments, a probability distribution of prices, which must be explained by an invariant probability distribution over the preferences in a sense that is explained in the next section.

4 The problem and definitions:

I assume that there is a finite set $\mathcal{I} = \{1, ..., I\}$ of consumers, and a finite number, $L \in \mathbb{N}$, of commodities. As usual, for each consumer $i \in \mathcal{I}$, the consumption set is the nonnegative orthant $\mathbb{R}^L_+$.

In order to avoid zero-degree homogeneity problems, I normalize prices to lie in the $(L-1)$-dimensional unit simplex, which I denote by $\mathcal{S}$. Since prices are going to be assumed as random, I assume that $\mathcal{S}$ is endowed with a σ-algebra Ξ. In empirical applications, Ξ is determined by the finesse with which prices are discerned.

I also assume that one observes a nonempty set, $E \subseteq \left(\mathbb{R}^L_{++}\right)^I$, of profiles of strictly positive endowments, and that for each observed profile $e = \left(e^i\right)_{i \in \mathcal{I}} \in E$, one observes a probability measure on the simplex of prices $\chi_e : \Xi \longrightarrow [0, 1]$. These elements constitute the whole of the observed data. Intuitively, given a discernible set of (normalized) prices $C \subseteq \mathcal{S}$, $C \in \Xi$, and given an observed profile of endowments, $e \in E$, the number $\chi_e(C) \in [0, 1]$ is to be understood as the frequency with which prices were observed to lie in C when endowments were e.

The question that I want to answer is when data constructed and interpreted as before are consistent with general equilibrium under random preferences. I must then first specify such definition of consistency and, in doing so, I must specify the class of preferences that I am going to allow myself to use when answering the question. Hence, let $\mathcal{U}$ be the family of all continuous, strongly concave, strictly monotone functions $U^i : \mathbb{R}^L_+ \longrightarrow \mathbb{R}$. Preferences that are representable by functions in the class $\mathcal{U}$ satisfy standard assumptions in economics and are useful in that they guarantee existence of general equilibrium, uniqueness of individual demand and Walras' law.

One would like to say that the data set $\left\{E, (\chi_e)_{e \in E}\right\}$ is rationalizable if the observed distributions of prices can be explained as being induced, as equilibrium prices, by a probability distribution over the set of profiles of preferences, $\mathcal{U}^I = (\mathcal{U})^I$. In this case, however, multiplicity of equilibria poses a serious problem as there is no reason to expect a one-to-one correspondence between preferences and Walrasian prices for given endowments. The standard assumption to make is then that prices are also determined randomly from within the Walras set of the economy. Since the latter set depends on the profile of endowments, one would be requiring that there exist, for each $e \in E$, a probability measure[5] $\pi_e : \mathcal{P}\left(\mathcal{U}^I\right) \times \Xi \longrightarrow [0,1]$ such that for each $C \in \Xi$, $\chi_e(C) = \pi_e\left(\mathcal{U}^I, C\right)$, subject to the constraint that for each $\mathcal{V} \in \mathcal{P}\left(\mathcal{U}^I\right)$ and each $C \in \Xi$,

$$\pi_e(\mathcal{V}, C) > 0 \Longrightarrow (\exists U \in \mathcal{V})(\exists p \in C) : p \in W_{U,e} \tag{1}$$

where for each $e \in \mathbb{R}^L_{++}$ and each $U = \left(U^i\right)_{i \in \mathcal{I}} \in \mathcal{U}^I$, the set of Walrasian equilibrium prices is denoted by $W_{U,e}$.

[5] For a set Z, I denote by $\mathcal{P}(Z)$ its power set, which is the set of all subsets of Z, and is therefore a σ-algebra on Z.

Intuitively, the first condition says that given any observed profile of endowments, $e \in E$, for any discernible set of prices $C \in \Xi$, the observed probability that prices lie in C is numerically explained by the theoretical joint distribution of preferences and prices, π_e, once all the possible profiles of preferences are taken into account (and, hence, "integrated out:" $\pi_e\left(\mathcal{U}^I, C\right)$. The condition, however, imposes by itself none of the principles of individual rationality and market clearing. This is done by the second condition, which requires that, given endowments $e \in E$, the theoretical joint probability assigned to a set of profiles of preferences $\mathcal{V} \in \mathcal{P}\left(\mathcal{U}^I\right)$ and a set of prices $C \in \Xi$ be positive only if for at least one of the profiles of preferences in $\mathcal{V}$ there is a price in C which is Walrasian equilibrium given the endowments e.

This, however, demands too little from the rationalization, because it in no sense requires independence in the random determination of preferences from the profiles of endowments, which one would like to have. In other words, under only these two conditions it could occur that the theoretical probabilities assigned to profiles of preferences depend on the profile of endowments, which would appear problematic and unsatisfactory unless one has a theory to explain such dependence. I then impose this independence condition to the rationalizing distribution, by requiring that the family $(\pi_e)_{e \in E}$ have a common marginal distribution over $\mathcal{U}^I$. That is to say, by demanding that there exist a probability measure $\vartheta : \mathcal{P}\left(\mathcal{U}^i\right) \longrightarrow [0,1]$, such that for every $\mathcal{V} \in \mathcal{P}\left(\mathcal{U}^I\right)$ and for every $e \in E$, $\vartheta(\mathcal{V}) = \pi_e(\mathcal{V}, \mathcal{S})$, where prices are integrated out. Then, each one of the conditional distributions $\pi_{e|U} : \Xi \longrightarrow [0,1]$, defined, for $U \in \mathcal{U}^I$ such that $\vartheta(\{U\}) > 0$ and $e \in E$, as

$$(\forall C \in \Xi) : \pi_{e|U}(C) = \frac{\pi_e(\{U\}, C)}{\vartheta(\{U\})} \tag{2}$$

makes sense on its own right, as they represent "random selectors" over $W_{U,e}$ as defined, for example, in Allen (1985). For notational simplicity, let me denote by $\digamma$ the set of all probability measures on $\mathcal{S}$, defined over Ξ. Given profiles of preferences $u \in \mathcal{U}^I$ and endowments $e \in \left(\mathbb{R}^L_{++}\right)^I$ a random selector is a function $\varphi \in \digamma$ such that $\varphi\left(W_{u,e}\right) = 1$.[6]

In order to distinguish sources of randomness, that is in order to distinguish randomness in preferences from the one that arises in prices even when preferences have been determined, I assume that there exists a nonempty and finite set, Ω, of "natural" states of the world, which account, only, for changes in the preferences of individuals. Rationalizability is then defined as:[7]

Definition 1 *A data set $\left\{E, (\chi_e)_{e\in E}\right\}$ is Ω-rationalizable if there exist a probability measure $\delta : \mathcal{P}(\Omega) \longrightarrow [0,1]$, a function $u : \Omega \longrightarrow \mathcal{U}^I$ and a function $\varphi : u[\Omega] \times E \longrightarrow \digamma$ such that for each $e \in E$ and each $C \in \Xi$:*[8]

$$\chi_e(C) = \sum_{\omega\in\Omega} \delta(\omega)\, \varphi(u(\omega), e)(C) \tag{3}$$

and for each $\omega \in \Omega$ and each $e \in E$:

$$\varphi(u(\omega), e)\left(W_{u(\omega),e}\right) = 1 \tag{4}$$

In the definition, δ is a probability distribution over the states of Nature, u is a rule that assigns, to each state of Nature, a profile of utilities for the individuals and φ assigns to each profile of utilities which can occur, $u(\omega)$,

[6] This is under the ideal assumption that $W_{u,e} \in \Xi$. A weaker requirement would be that

$$(\forall C \in \Xi) : W_{u,e} \subseteq C \Longrightarrow \varphi(C) = 1$$

Given assumptions that I will introduce below, I can work under the "ideal" assumption.

[7] Given a function $f : X \longrightarrow Y$ and a set $Z \subseteq X$, $f[Z]$ is the image of Z under f:

$$f[Z] = \{y \in Y\ y \in Y | (\exists x \in Z) : f(x) = y\}$$

[8] Notation is somewhat tricky here. Since φ maps $u[\Omega] \times E$ into $\digamma$, which is a function space, then, for $u \in u[\Omega]$ and $e \in E$, $\varphi(u,e) \in \digamma$ means that $\varphi(u,e)$ is a function with domain Ξ and target set $[0,1]$. For $C \in \Xi$, the value of this function is $\varphi(u,e)(C)$

and each profile of endowments e, a probability distribution over $\mathcal{S}$. The first condition in the definition is again that the observed probabilities be explained by δ, u and φ, whereas the second one requires that each distribution over $\mathcal{S}$ have as support the Walras set of its economy, so that it be a bona fide random selector.

In order to solve the problem, I will consider only the finite case with fully fine discernibility of prices. That is, throughout the rest of the paper I will maintain the assumptions that $\Xi = \mathcal{P}(\mathcal{S})$, $\#E < \infty$ and[9]

$$(\forall e \in E) : \#Supp\,(\chi_e) < \infty \wedge \chi_e\,(Supp\,(\chi_e)) = 1 \tag{5}$$

where

$$Supp\,(\chi_e) = \{p \in \mathcal{S} |\, \chi_e\,(\{p\}) > 0\} \subseteq \mathbb{R}^L_{++} \tag{6}$$

That is, I assume that prices can be observed with infinite accuracy, that only a finite number of profiles of endowments have been observed (as in Brown and Matzkin, 1996) and that for each observed profile of endowments only a finite number of strictly positive prices have been observed to occur.[10]

5 Rationalizable data sets:

In Brown and Matzkin (1996), given observed endowments and prices, the existence of unobservable individual preferences was implicitly tested through the existence of individual demands (for each endowment and price) which were

[9] The logical connector 'and' will be denoted by $\wedge$. The symbol $\vee$ will denote 'or', while $\neg$ will denote negation of a subsequent logical sentence. Parenthesis will be used to clarify sentences and their quantification. The existencial and universal quantifiers will be given their standard notation, $\exists$ and $\forall$ respectively.

[10] Under the null hypothesis of rationalizability, if one restricts $\mathcal{U}$ to include only differentiable functions with interior contours, this assumption holds generically on endowments. This follows, given that Ω is finite, from Debreu (1970), since demands must in this case be continuously differentiable, as shown by Debreu (1972). In this case, only a finite number of prices can have positive probability, which I assume by imposing that all the observed distributions have finite, and hence discrete, support.

required to satisfy the Strong Axiom of Revealed Preferences (SARP), Walras' law and market clearing.

Here, one observes individual endowments $e \in E$, the supports of each of the distributions of prices given those endowments $\{Supp(\chi_e) \subseteq \mathcal{S}\}_{e \in E}$, and the actual probabilities that each one of the prices in these supports attains

$$\{\chi_e(\{p\})\}_{p \in Supp(\chi_e), e \in E} \tag{7}$$

The unobservables whose existence one wants to establish are the profiles of preferences that occur in each state of the world $\{u(\omega)\}_{\omega \in \Omega}$, the probability that each state of the world attains, $\{\delta(\omega)\}_{\omega \in \Omega}$, and the random selectors given utilities and endowments $\{\varphi(u(\omega), e)\}_{\omega \in \Omega, e \in E}$.

There exists, however, a third class of variables: those that one could observe under the ideal assumption of being able to access individual-level data, as if one could use the economy as an experimental laboratory. There are two groups of data in this set of "observable-but-unobserved" variables. The first ones are the demands of each individual, for each one of the budgets that he has ever faced, and for each one of the states of nature.[11] Second, since for each profile of individual budgets actual choices depend on ω, then the profiles of demands from those budgets are also random variables. If one had access to individual-level data, one would know the distributions of these variables. Hence, the probability distributions of these choices are in this category of observable-but-unobserved.

If these two groups of variables were observed, under the null hypothesis of consistency with general equilibrium they would have to satisfy certain conditions implied by individual rationality under random preferences and market clearing. For the first group of variables, the necessary and sufficient conditions are that for every individual and given a state of nature, demands must satisfy

[11] Under the null hypothesis, some of these exercises may be clearly counterfactual.

SARP across budgets. For the second type of variables, I use the extension of the Axiom of Stochastic Revealed Preference (ASRP), originally proposed by McFadden and Richter (1990) and extended by Carvajal (2002b) to the case of collective choices over not-necessarily-finite choice sets. Finally, if one could also observe random selectors, and they were therefore treated as observable-but-unobserved, then the restrictions of their definition, namely that they be probability measures and have supports within the Walras sets, should be imposed directly.

Theorems 1 and 2 below characterize rationalizability in terms of existence of these observable-but-unobserved variables and the aforementioned conditions. Before they can be stated, I need to introduce the following pieces of notation.

Notation 1 *Given* $\left\{E, (\chi_e)_{e \in E}\right\}$, *denote, for each* $i \in \mathcal{I}$*:*

$$\mathcal{B}^i = \left\{ B^i \subseteq \mathbb{R}^L_+ \middle| \, (\exists e \in E) \, (\exists p \in Supp\,(\chi_e)) : B\left(p, e^i\right) = B^i \right\} \tag{8}$$

Denote also:

$$\mathcal{B} = \left\{ B \subseteq \left(\mathbb{R}^L_+\right)^I \middle| \, (\exists e \in E) \, (\exists p \in Supp\,(\chi_e)) : B = \prod_{i \in \mathcal{I}} B\left(p, e^i\right) \right\} \tag{9}$$

where for $p \in \mathcal{S}$ *and* $e^i \in \mathbb{R}^L_+$,

$$B\left(p, e^i\right) = \left\{ x \in \mathbb{R}^L_+ \middle| \, p \cdot x \leqslant p \cdot e^i \right\} \tag{10}$$

Notation 2 *Suppose that for each* $i \in \mathcal{I}$, $\Gamma^{i,B^i} \subseteq \mathcal{P}\left(B^i\right) \setminus \{\varnothing\}$. *Given* $B = \prod_{i \in \mathcal{I}} B^i \in \mathcal{B}$, *denote*

$$\Gamma^B = \left\{ C \subseteq B | \left(\exists \left(C^i\right)_{i \in \mathcal{I}} \in \prod_{i \in \mathcal{I}} \Gamma^{i,B^i} \right) : \prod_{i \in \mathcal{I}} C^i = C \right\} \tag{11}$$

and denote by Σ^B the σ-algebra generated by Γ^B on B. Moreover, denote:

$$\mathcal{B} \otimes \Sigma = \bigcup_{B \in \mathcal{B}} \left(\{B\} \times \Sigma^B \right) \tag{12}$$

Notation 3 *For any set $Z \subseteq \left(\mathbb{R}^L\right)^I$, denote its indicator function by $1_Z : \left(\mathbb{R}^L\right)^I \longrightarrow \{0,1\}$, which is defined by*

$$\left(\forall x \in \left(\mathbb{R}^L\right)^I\right) : 1_Z(x) = \begin{cases} 1 & \text{if } x \in Z \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

Using this notation[12], the first characterization of rationalizability is given next. Its importance is that it disentangles all the mechanisms that are present behind a rationalizable data set.

Theorem 1 *A data set $\left\{E, (\chi_e)_{e \in E}\right\}$ is Ω-rationalizable only if*

- *For each $i \in \mathcal{I}$, each $B^i \in \mathcal{B}^i$ and each $\omega \in \Omega$, there exist $x^{i,B^i,\omega} \in \mathbb{R}^L_+$;*
- *Defining, for each $i \in \mathcal{I}$ and each $B^i \in \mathcal{B}^i$,*

$$\Gamma^{i,B^i} = \bigcup_{\omega \in \Omega} \left\{ \left\{ x^{i,B^i,\omega} \right\} \right\} \tag{14}$$

  then, for each $B \in \mathcal{B}$ and each $C \in \Sigma^B$ there exists $g_{B,C} \in \mathbb{R}_+$;

- *For each $\omega \in \Omega$, there exists $d_\omega \in \mathbb{R}_+$;*
- *For each $\omega \in \Omega$, each $e \in E$ and each $p \in Supp(\chi_e)$, there exists $f_{\omega,e,p} \in \mathbb{R}$*

which satisfy the following conditions:

[12] And, also, for any set Z and any $K \in \mathbb{N} \cup \{\infty\}$, denoting by $\{z_k\}_{k=1}^K \overset{seq}{\subseteq} Z$ the fact that $\{z_k\}_{k=1}^K$ is a sequence defined in Z.

1. For each $i \in \mathcal{I}$, each $\omega \in \Omega$, each $K \in \mathbb{N}$ and each $\left\{B_k^i\right\}_{k=1}^K \overset{seq}{\subseteq} \mathcal{B}^i$:

$$\left((\forall k \in \{1, ..., K-1\}) : x^{i,B_{k+1}^i,\omega} \in B_k^i\right) \implies \begin{pmatrix} x^{i,B_1^i,\omega} = x^{i,B_K^i,\omega} \\ \vee \\ x^{i,B_1^i,\omega} \notin B_K^i \end{pmatrix} \tag{15}$$

2. For each $i \in \mathcal{I}$, each $\omega \in \Omega$, each $e \in E$ and each $p \in Supp\,(\chi_e)$:

$$p \cdot x^{i,B\left(p,e^i\right),\omega} = p \cdot e^i \tag{16}$$

3. For each $B \in \mathcal{B}$,

$$g_{B,B} = 1 \tag{17}$$

and for each $B \in \mathcal{B}$, each $K \in \mathbb{N}$ and each disjoint $\{C_k\}_{k=1}^K \overset{seq}{\subseteq} \Sigma^B$:

$$\sum_{k=1}^K g_{B,C_k} = g_{B,\cup_{k=1}^K C_k} \tag{18}$$

4. For each $K \in \mathbb{N}$ and each $\{B_k, C_k\} \overset{seq}{\subseteq} \mathcal{B} \otimes \Sigma$, there exists $\omega \in \Omega$ such that

$$\sum_{k=1}^K g_{B_k,C_k} \leqslant \sum_{k=1}^K \mathbf{1}_{C_k}\left(\left(x^{i,B_k^i,\omega}\right)_{i\in\mathcal{I}}\right) \tag{19}$$

5. For each $B \in \mathcal{B}$ and each $C \in \Sigma^B$

$$g_{B,C} = \sum_{\omega\in\Omega} d_\omega \mathbf{1}_C\left(\left(x^{i,B^i,\omega}\right)_{i\in\mathcal{I}}\right) \tag{20}$$

6. *For each $e \in E$, each $p \in Supp(\chi_e)$ and each $\omega \in \Omega$:*

$$\sum_{i \in \mathcal{I}} x^{i, B\left(p, e^i\right), \omega} \neq \sum_{i \in \mathcal{I}} e^i \Longrightarrow f_{\omega, e, p} = 0 \tag{21}$$

7. *For each $\omega \in \Omega$ and each $e \in E$:*

$$d_w > 0 \Longrightarrow \sum_{p \in Supp(\chi_e)} f_{\omega, e, p} = 1 \tag{22}$$

8. *For each $e \in E$ and each $p \in Supp(\chi_e)$:*

$$\chi_e(\{p\}) = \sum_{\omega \in \Omega} d_\omega f_{\omega, e, p} \tag{23}$$

Proof. Suppose that $\delta : \mathcal{P}(\Omega) \longrightarrow [0,1]$, $u : \Omega \longrightarrow (\mathcal{U})^I$ and $\varphi : u[\Omega] \longrightarrow F$ Ω-rationalize $\left\{E, (\chi_e)_{e \in E}\right\}$. Denoting, $\forall \omega \in \Omega$, $u(\omega) = \left(u^i(\omega)\right)_{i \in \mathcal{I}}$, implicitly define $\forall i \in \mathcal{I}$, $\forall B^i \in \mathcal{B}^i$ and $\forall \omega \in \Omega$, $x^{i, B^i, \omega} \in \mathbb{R}_+^L$ by[13]

$$\left\{x^{i, B^i, \omega}\right\} = Arg \max_{y \in B^i} u^i(\omega)(y) \tag{24}$$

which one can do since each B^i is nonempty and compact and each $u^i(\omega)$ is continuous and strongly concave. Denote:

$$x^{i, B^i, \omega} = \arg \max_{y \in B^i} u^i(\omega)(y) \tag{25}$$

Then, $\forall i \in \mathcal{I}$ and $\forall \omega \in \Omega$, Theorem 2 in Matzkin and Richter (1991), (b)$\Longrightarrow$(a) implies condition *1*, whereas condition *2* follows, by construction, from the monotonicity of $u^i(\omega)$.

[13] The notational proviso of note 8 applies here: $u^i(\omega)$ is a function mapping $\mathbb{R}_+^L$ into $\mathbb{R}$, which takes the value $u^i(\omega)(y)$ at $y \in \mathbb{R}_+^L$.

Define $\forall B \in \mathcal{B}$ and $\forall C \in \Sigma^B$,

$$g_{B,C} = \delta\left(\left\{\omega \in \Omega | \left(\arg\max_{y \in B^i} u^i(\omega)(y)\right)_{i \in \mathcal{I}} \in C\right\}\right) \tag{26}$$

Condition *3* is immediate, whereas condition *4* follows from theorem 1 in Carvajal (2002b).[14]

Define now, $\forall \omega \in \Omega$, $d_\omega = \delta(\{\omega\})$.Then, by construction,

$$\begin{aligned}
g_{B,C} &= \delta\left(\left\{\omega \in \Omega | \left(\arg\max_{y \in B^i} u^i(\omega)(y)\right)_{i \in \mathcal{I}} \in C\right\}\right) \\
&= \delta\left(\left\{\omega \in \Omega | \left(x^{i,B^i,\omega}\right)_{i \in \mathcal{I}} \in C\right\}\right) \\
&= \sum_{\omega \in \Omega : \left(x^{i,B^i,\omega}\right)_{i \in \mathcal{I}} \in C} \delta(\{\omega\}) \qquad (27) \\
&= \sum_{\omega \in \Omega} \delta(\{\omega\}) \mathbf{1}_C\left(\left(x^{i,B^i,\omega}\right)_{i \in \mathcal{I}}\right) \\
&= \sum_{\omega \in \Omega} d_\omega \mathbf{1}_C\left(\left(x^{i,B^i,\omega}\right)_{i \in \mathcal{I}}\right)
\end{aligned}$$

which is condition *5*.

Define $\forall \omega \in> \Omega$, $\forall e \in E$ and $\forall p \in Supp(\chi_e)$, $f_{\omega,e,p} = \varphi(u(\omega), e)(\{p\})$. Also, define $\forall \omega \in \Omega$, $\forall e \in E$ and $\forall p \in Supp(\chi_e)$, $f_{\omega,e,p} = \varphi(u(\omega), e)(\{p\})$ and suppose that for $\omega \in \Omega$, $e \in E$ and $p \in Supp(\chi_e)$,

$$\sum_{i \in \mathcal{I}} x^{i,B(p,e^i),\omega} \neq \sum_{i \in \mathcal{I}} e^i \tag{28}$$

[14] Since $\#\Omega < \infty$, it follows that $\forall i \in \mathcal{I}$ and $\forall B^i \in \mathcal{B}^i$, $\#\Gamma^{i,B^i} < \infty$, and hence, $\forall B \in \mathcal{B}$, $\#\Gamma^B < \infty$. Since Σ^B is the σ-algebra generated by Γ^B, it is true that $\#\Sigma^B < \infty$. Since $\#E < \infty$, and $\forall e \in E$, $\#Supp(\chi_e) < \infty$ then $\#\mathcal{B} < \infty$ and therefore condition 1 in Carvajal (2002b) is satisfied. Conditions 2 and 3 are also satisfied since $\forall B \in \mathcal{B}$, $B = \prod_{i \in \mathcal{I}} \overrightarrow{B}^i = \prod_{i \in \mathcal{I}} B^i$. Alternatively, the result can be argued using theorem 2 in Carvajal (2002b), given remark 2 there.

By construction, this means that

$$\sum_{i\in\mathcal{I}}\arg\max_{y\in B(p,e^i)}u^i(\omega)(y)\neq\sum_{i\in\mathcal{I}}e^i \tag{29}$$

which implies that $p\notin W_{u(\omega),e}$ and, then, since $\varphi(u(\omega),e)\left(W_{u(\omega),e}\right)=1$, it follows that $\varphi(u(\omega),e)(\{p\})=0$ and, therefore, that $f_{\omega,e,p}=0$, implying condition *6*.

Now, let $\widetilde{\omega}\in\Omega$ and $e\in E$ and suppose that $d_{\widetilde{\omega}}>0$ and $\sum_{p\in Supp(\chi_e)}f_{\omega,e,p}\neq 1$. By construction, since $\varphi(u(\widetilde{\omega}),e)\in\digamma$, it must be that

$$\sum_{p\in Supp(\chi_e)}\varphi(u(\widetilde{\omega}),e)(\{p\})<1 \tag{30}$$

which implies that $\exists C\subseteq\mathcal{S}\backslash Supp(\chi_e):\varphi(u(\widetilde{\omega}),e)(C)>0$. Then, since $\delta(\widetilde{\omega})=d_{\widetilde{\omega}}>0$, it follows that

$$\begin{aligned}\chi_e(C) &= \sum_{\omega\in\Omega}\delta(\omega)\varphi(u(\omega),e)(C)\\ &\geqslant \delta(\widetilde{\omega})\varphi(u(\widetilde{\omega}),e)(C)\\ &> 0\end{aligned} \tag{31}$$

contradicting the fact that $C\subseteq\mathcal{S}\backslash Supp(\chi_e)$. This implies condition *7*.

Finally, by construction, $\forall e\in E$ and $\forall p\in Supp(\{p\})$,

$$\begin{aligned}\chi_e(\{p\}) &= \sum_{\omega\in\Omega}\delta(\omega)\varphi(u(\omega),e)(\{p\})\\ &= \sum_{\omega\in\Omega}d_\omega f_{\omega,e,p}\end{aligned} \tag{32}$$

which is condition *8*. ■

At the risk of being pedantic, let me give intuition about the conditions of the theorem. Suppose that the null hypothesis of Ω-rationalizability, for some set Ω of events, is true. Then,

- For each individual $i \in \mathcal{I}$, each budget $B^i \in \mathcal{B}^i$, and each state of the world $\omega \in \Omega$, let $x^{i,B^i,\omega} \in \mathbb{R}_+^L$ be i's utility-maximizing demand over B^i when ω realizes.

- For each individual $i \in \mathcal{I}$ and each budget $B^i \in \mathcal{B}^i$, let $\Gamma^{i,B^i} \subseteq \mathbb{R}_+^L$ be the collection of (singleton sets of) bundles that constitute i's utility maximizing demands over B^i, considering all possible events in Ω.

- For each collective budget $B = \prod_{i\in\mathcal{I}} B^i \in \mathcal{B}^i$, let Σ^B be the product σ-algebra generated by the collection $\left\{\Gamma^{i,B^i}\right\}_{i\in\mathcal{I}}$ over B.

- For each collective budget $B = \prod_{i\in\mathcal{I}} B^i \in \mathcal{B}^i$ and each set of profiles of bundles $C \in \Sigma^B$, let $g_{B,C} \in \mathbb{R}_+$ be the probability that if each individual chooses from B^i individually rationally, then the profile of choices lies in C.

- For each $\omega \in \Omega$, let d_ω be its probability.

- For each state of the world $\omega \in \Omega$, each profile of endowments $e \in E$ and each observed price (given e) $p \in \mathit{Supp}\left(\chi_e\right)$, let $f_{\omega,e,p} \in \mathbb{R}_+$ be the probability assigned to price p by the random selector corresponding to e and the profile of preferences assigned at ω.

Under this interpretation of the variables used in the theorem, the intuition of its conditions is as follows:

- Fix an individual $i \in \mathcal{I}$ and a state of the world $\omega \in \Omega$. By doing so, one is also fixing preferences which are represented by $u^i\left(\omega\right)$. Individual rationality then imposes that the SARP be satisfied across all possible budgets, which is condition *1*.

- Fix an individual $i \in \mathcal{I}$, a state of the world $\omega \in \Omega$ and a budget $B^i \in \mathcal{B}^i$. Since i's preferences in ω are strictly monotonic, Walras' law must be satisfied, which is condition *2*

- Fix a collective budget $B \in \mathcal{B}$. Condition *3* is a straightforward application of Kolmogorov's axioms given the definition of the numbers $\{g_{B,C}\}_{C \in \Sigma^B}$ as probabilities.

- Condition *4* is less straightforward. It is an application of the extension of the Axiom of Stochastic Revealed Preference of McFadden and Richter (1990) to collective problems as proposed by Carvajal (2002b). Its intuition is that events that are likely to happen should happen often. That is, recall that for each $(B, C) \in \mathcal{B} \otimes \Sigma$, the number $g_{B,C}$ is the probability that the profile of individually-rational choices from B lies in C, and that for given state of the world $\omega \in \Omega$ such profile of choices is $\left(x^{i,B^i,\omega}\right)_{i \in \mathcal{I}}$. Now, for a sequence of collective budgets and collective choice sets $\{B_k, C_k\}_{k=1}^K \overset{seq}{\subseteq} \mathcal{B} \otimes \Sigma$, consider the event 'for each k, choosing rationally from B_k^i, individuals determine a profile of choices that lies in C_k.' If such an event occurs with "high" probability, in the sense that the number $\sum_{k=1}^K g_{B_k,C_k}$ is "high," then, it should also be true that for at least one state of the world $\omega \in \Omega$ it happens that for "many" of the collective budgets B_k their profile of individually-rational choices lies in C_k, so that the number

$$\sum_{k=1}^{K} \mathbf{1}_{C_k} \left(\left(x^{i,B_k^i,\omega} \right)_{i \in \mathcal{I}} \right) \tag{33}$$

is also "high."

- Condition *5* specifies that, indeed, the probabilities over collective budgets $g_{B,C}$ are explained by the probabilities over states of the world, via individual rationality.

- Fix a profile of endowments $e \in E$, an observed price $p \in Supp\,(\chi_e)$ and a state of the world $\omega \in \Omega$. If the profile of individually-rational demands, $\left(x^{i,B\left(p,e^i\right),\omega}\right)_{i\in\mathcal{I}}$, does not clear markets, then it cannot be assigned a positive probability by the random selector at e and the profile of preferences in ω. This is imposed by condition *6*.

- Fix a state of the world $\omega \in \Omega$. Condition *6* has restricted the supports of the random selectors to Walras' sets. Condition *7* simply implies that they are, indeed, probability distributions.

- Given Ω-rationalizability, probabilities over states of the world and random selectors must explain the observed probabilities accurately. This is precisely condition *8*.

If collective budgets and states of the world are fully discriminating in terms of individual behavior, in the sense that for each state of the world there exists a budget for which individual behavior would differ from all the rest of states of the world, then one does not need probabilities both over states of the world and over collective choices, and just the latter suffices for Ω-rationalizability, as theorem 2 shows. The intuition of its conditions is the same as on theorem 1, only that by strengthening condition *5* one can now discern probabilities over states of the world from those over collective choices.

Theorem 2 *Given Ω, suppose that a data set $\left\{E, (\chi_e)_{e\in E}\right\}$ satisfies that*

- *For each $i \in \mathcal{I}$, each $B^i \in \mathcal{B}^i$ and each $\omega \in \Omega$, there exist $x^{i,B^i,\omega} \in \mathbb{R}^L_+$;*

- *Defining, for each $i \in \mathcal{I}$ and each $B^i \in \mathcal{B}^i$,*

$$\Gamma^{i,B^i} = \bigcup_{\omega\in\Omega} \left\{\left\{x^{i,B^i,\omega}\right\}\right\} \tag{34}$$

then, for each $B \in \mathcal{B}$ *and each* $C \in \Sigma^B$ *there exists* $g_{B,C} \in \mathbb{R}_+$*;*

- *For each* $\omega \in \Omega$*, each* $e \in E$ *and each* $p \in Supp\,(\chi_e)$*, there exists* $f_{\omega,e,p} \in \mathbb{R}_+$*,*

which satisfy the following conditions:

1. *For each* $i \in \mathcal{I}$*, each* $\omega \in \Omega$*, each* $K \in \mathbb{N}$ *and each* $\left\{B_k^i\right\}_{k=1}^K \overset{seq}{\subseteq} \mathcal{B}^i$*:*

$$\begin{aligned}\Big((\forall k \in \{1, ..., K-1\}) : x^{i,B_{k+1}^i,\omega} \in B_k^i\Big) \\ \Longrightarrow \begin{pmatrix} x^{i,B_1^i,\omega} = x^{i,B_K^i,\omega} \\ \vee \\ x^{i,B_1^i,\omega} \notin B_K^i \end{pmatrix}\end{aligned} \tag{35}$$

2. *For each* $i \in \mathcal{I}$*, each* $\omega \in \Omega$*, each* $e \in E$ *and each* $p \in Supp\,(\chi_e)$*:*

$$p \cdot x^{i,B\left(p,e^i\right),\omega} = p \cdot e^i \tag{36}$$

3. *For each* $B \in \mathcal{B}$*,*

$$g_{B,B} = 1 \tag{37}$$

and for each $B \in \mathcal{B}$*, each* $K \in \mathbb{N}$ *and each disjoint* $\{C_k\}_{k=1}^K \overset{seq}{\subseteq} \Sigma^B$*:*

$$\sum_{k=1}^{K} g_{B,C_k} = g_{B,\cup_{k=1}^K C_k} \tag{38}$$

4. *For each* $K \in \mathbb{N}$ *and each* $\{B_k, C_k\} \overset{seq}{\subseteq} \mathcal{B} \otimes \Sigma$*, there exists* $\omega \in \Omega$ *such that*

$$\sum_{k=1}^{K} g_{B_k,C_k} \leqslant \sum_{k=1}^{K} \mathbf{1}_{C_k}\left(\left(x^{i,B_k^i,\omega}\right)_{i \in \mathcal{I}}\right) \tag{39}$$

5.

$$\left(\forall\omega\in\Omega\right)\left(\exists B\in\mathcal{B}\right)\left(\forall\widetilde{\omega}\in\Omega\backslash\left\{\omega\right\}\right)\left(\exists i\in\mathcal{I}\right):x^{i,B^{i},\widetilde{\omega}}\neq x^{i,B^{i},\omega}\tag{40}$$

6. For each $e\in E$, each $p\in Supp\left(\chi_{e}\right)$ and each $\omega\in\Omega$:

$$\sum_{i\in\mathcal{I}}x^{i,B\left(p,e^{i}\right),\omega}\neq\sum_{i\in\mathcal{I}}e^{i}\Longrightarrow f_{\omega,e,p}=0\tag{41}$$

7. For each $\omega\in\Omega$ such that

$$g_{B(\omega),\left\{\left(x^{i,B^{i}(\omega),\omega}\right)_{i\in\mathcal{I}}\right\}}>0\tag{42}$$

and for each $e\in E$:

$$\sum_{p\in Supp(\chi_{e})}f_{\omega,e,p}=1\tag{43}$$

8. For each $e\in E$ and each $p\in Supp\left(\chi_{e}\right)$:

$$\chi_{e}\left(\left\{p\right\}\right)=\sum_{\omega\in\Omega}g_{B(\omega),\left\{\left(x^{i,B^{i}(\omega),\omega}\right)_{i\in\mathcal{I}}\right\}}f_{\omega,e,p}\tag{44}$$

Where $B\left(\omega\right)\in\mathcal{B}$ is implicitly defined by

$$B=B\left(\omega\right)\Longleftrightarrow\left(\forall\widetilde{\omega}\in\Omega\backslash\left\{\omega\right\}\right)\left(\exists i\in\mathcal{I}\right):x^{i,B^{i},\widetilde{\omega}}\neq x^{i,B^{i},\omega}\tag{45}$$

Then, $\left\{E,\left(\chi_{e}\right)_{e\in E}\right\}$ is Ω-rationalizable.

Proof. Given conditions *1* and *2*, it follows from theorem 2 in Matzkin and Richter (1991), (a)$\Longrightarrow$(b), that $\forall i\in\mathcal{I}$ and $\forall\omega\in\Omega$, $\exists U^{i,\omega}\in\mathcal{U}$ such that

$$\left(\forall B^{i}\in\mathcal{B}^{i}\right):Arg\max_{y\in B^{i}}U^{i,\omega}\left(y\right)=\left\{x^{i,B^{i},\omega}\right\}\tag{46}$$

Define the function $u : \Omega \longrightarrow \mathcal{U}^I$ by $(\forall \omega \in \Omega) : u(\omega) = \left(U^{i,\omega}\right)_{i \in \mathcal{I}}$

Now, recalling that $\forall i \in \mathcal{I}$, and $\forall B^i \in \mathcal{B}^i$:

$$\Gamma^{i,B^i} = \bigcup_{\omega \in \Omega} \left\{ x^{i,B^i,\omega} \right\} \tag{47}$$

and defining $\forall B \in \mathcal{B}$, the function $\gamma_B : \Sigma^B \longrightarrow [0,1]$ by $\left(\forall C \in \Sigma^B\right) : \gamma_B(C) = g_{B,C}$, it follows from condition *3* that γ_B is a probability measure and then, from condition *4* and theorem 1 in Carvajal (2002b), that $\exists \delta : \mathcal{P}(\Omega) \longrightarrow [0,1]$, a probability measure, such that $\forall B \in \mathcal{B}$ and $\forall C \in \Sigma^B$:

$$\delta\left(\left\{\omega \in \Omega | \left(\arg\max_{y \in B^i} u^i(\omega)(y)\right)_{i \in \mathcal{I}} \in C\right\}\right) = g_{B,C} \tag{48}$$

Define also $\forall \omega \in \Omega$, $B(\omega)$ as

$$B(\omega) = B \in \mathcal{B} \Longleftrightarrow (\forall \widetilde{\omega} \in \Omega \backslash \{\omega\}) (\exists i \in \mathcal{I}) : x^{i,B^i,\widetilde{\omega}} \neq x^{i,B^i,\omega} \tag{49}$$

which one can do by condition *5*. By construction, $\forall \omega \in \Omega :$

$$g_{B(\omega),\left\{\left(x^{i,B^i(\omega),\omega}\right)_{i \in \mathcal{I}}\right\}} = \gamma_{B(\omega)}\left(\left\{\left(x^{i,B^i(\omega),\omega}\right)_{i \in \mathcal{I}}\right\}\right) \tag{50}$$

whereas

$$\begin{aligned} & \gamma_{B(\omega)}\left(\left\{\left(x^{i,B^i(\omega),\omega}\right)_{i \in \mathcal{I}}\right\}\right) \\ = \quad & \delta\left(\left\{\widetilde{\omega} \in \Omega | \left(\arg\max_{y \in B^i(\omega)} u^i(\widetilde{\omega})(y)\right)_{i \in \mathcal{I}} \in \left\{\left(x^{i,B^i(\omega),\omega}\right)_{i \in \mathcal{I}}\right\}\right\}\right) \\ = \quad & \delta\left(\left\{\widetilde{\omega} \in \Omega | (\forall i \in \mathcal{I}) : x^{i,B^i(\widetilde{\omega}),\omega} = x^{i,B^i(\omega),\omega}\right\}\right) \\ = \quad & \delta(\{\omega\}) \end{aligned} \tag{51}$$

where the last step follows by definition of $B(\omega)$.

Now, construct $\varphi : u[\Omega] \times E \longrightarrow \digamma$ as follows. Let $u \in u[\Omega]$ and let $e \in E$. By definition and condition *5*, $\#\{\omega \in \Omega | \, u(\omega) = u\} = 1$. Then, let $\{\omega_u\} = \{\omega \in \Omega | \, u(\omega) = \omega\}$.

If $g_{B(\omega_u),\left\{\left(x^{i,B^i(\omega_u),\omega_u}\right)_{i\in\mathcal{I}}\right\}} > 0$, then define $\varphi(u,e) : \Xi \longrightarrow [0,1]$ as follows:

$$\begin{aligned} (\forall p \in Supp(\chi_e)) \quad &: \quad \varphi(u,e)(\{p\}) = f_{\omega_u,e,p} \\ (\forall p \in \mathcal{S} \backslash Supp(\chi_e)) \quad &: \quad \varphi(u,e)(\{p\}) = 0 \\ (\forall D \in \Xi : \#D \neq 1) \quad &: \quad \varphi(u,e)(D) = \sum_{p\in D} \varphi(u,e)(\{p\}) \end{aligned} \tag{52}$$

Condition *7* implies that, so defined, $\varphi(u,e) \in \digamma$.

If, alternatively, $g_{B(\omega_u),\left\{\left(x^{i,B^i(\omega_u),\omega_u}\right)_{i\in\mathcal{I}}\right\}} = 0$, then let $p \in W_{u,e}$, which exists by Arrow and Debreu (1954) and define $\varphi(u,e) : \Xi \longrightarrow [0,1]$ by:

$$(\forall D \in \Xi) : \varphi(u,e)(D) = \begin{cases} 1 & \text{if } \; p \in D \\ 0 & \text{otherwise} \end{cases} \tag{53}$$

I now claim that $u : \Omega \longrightarrow \mathcal{U}^I$, $\delta : \mathcal{P}(\Omega) \longrightarrow [0,1]$ and $\varphi : u[\Omega] \times E \longrightarrow \digamma$ Ω-rationalize $\left\{E, (\chi_e)_{e\in E}\right\}$.

First, let $e \in E$ and $p \in Supp(\chi_e)$. Then,

$$\begin{aligned} \sum_{\omega\in\Omega} \delta(\omega)\, \varphi(u(\omega), e)(\{p\}) \quad &= \quad \sum_{\omega\in\Omega} g_{B(\omega),\left\{\left(x^{i,B^i(\omega),\omega}\right)_{i\in\mathcal{I}}\right\}} f_{\omega,e,p} \\ &= \quad \chi_e(\{p\}) \end{aligned} \tag{54}$$

where I have used condition *8*. Hence $\forall e \in E$ and $\forall C \in \Xi$:

$$\begin{aligned}
\chi_e(C) &= \chi_e(C \cap Supp(\chi_e)) \\
&= \sum_{p \in C \cap Supp(\chi_e)} \chi_e(\{p\}) \\
&= \sum_{p \in C \cap Supp(\chi_e)} \sum_{\omega \in \Omega} \delta(\omega) \varphi(u(\omega), e)(\{p\}) \\
&= \sum_{p \in C \cap Supp(\chi_e)} \sum_{\omega \in \Omega : \delta(\{\omega\})>0} \delta(\omega) \varphi(u(\omega), e)(\{p\}) \qquad (55) \\
&= \sum_{\omega \in \Omega : \delta(\{\omega\})>0} \left(\delta(\omega) \sum_{p \in C \cap Supp(\chi_e)} \varphi(u(\omega), e)(\{p\}) \right) \\
&= \sum_{\omega \in \Omega : \delta(\{\omega\})>0} \left(\delta(\omega) \sum_{p \in C} \varphi(u(\omega), e)(\{p\}) \right) \\
&= \sum_{\omega \in \Omega : \delta(\{\omega\})>0} \delta(\omega) \varphi(u(\omega), e)(C) \\
&= \sum_{\omega \in \Omega} \delta(\omega) \varphi(u(\omega), e)(C)
\end{aligned}$$

where the sixth step follows from the fact that, by condition *7*, $\forall \omega \in \Omega$ such that $\delta(\{\omega\}) > 0$, if $p \notin Supp(\chi_e)$ then $\varphi(u(\omega), e)(\{p\}) = 0$.

Second, fix $\omega \in \Omega$ such that $g_{B(\omega), \left\{ \left(x^{i, B^i(\omega), \omega} \right)_{i \in \mathcal{I}} \right\}} > 0$ and let $e \in E$. Suppose that $p \notin W_{u(\omega),e}$. If $p \notin Supp(\chi_e)$, it follows by construction that $\varphi(u(\omega), e)(\{p\}) = 0$. Now, if $p \in Supp(\chi_e) \backslash W_{u(\omega),e}$, it follows that

$$\sum_{i \in \mathcal{I}} x^{i, B\left(p, e^i\right), \omega} \neq \sum_{i \in \mathcal{I}} e^i \qquad (56)$$

and hence, from condition *6*, one has that

$$\begin{aligned}
\varphi(u(\omega), e)(\{p\}) &= f_{\omega, e, p} \qquad (57) \\
&= 0
\end{aligned}$$

It then follows that $\varphi\left(u\left(\omega\right),e\right)\left(\mathcal{S}\backslash W_{u(\omega),e}\right)=0$ and that $\varphi\left(u\left(\omega\right),e\right)\left(W_{u(\omega),e}\right)=1$.

That the same conclusion applies $\forall\omega\in\Omega$ such that $g_{B(\omega),\left\{\left(x^{i,B^{i}(\omega),\omega}\right)_{i\in\mathcal{I}}\right\}}=0$ is given by construction. ■

6 Nonrationalizable data sets:

The results of the previous section characterizes rationalizability via the existence of several unobserved variables satisfying certain conditions. It could happen, however, that values for those variables satisfying such conditions always exist, so that the general equilibrium hypothesis is irrefutable. I will show in section 7 that one can obtain testable conditions purely on the observed data. However, since I will not obtain these quantifier-free restrictions, but will only show their existence, the question would still arise of whether or not they are tautological. I now show examples of nonrationalizable data sets, whose existence implies that the quantifier-free restrictions are not tautological and that the hypothesis is refutable.

There are two types of examples, which correspond to the existence or not of particular observable-but-unobserved variables. The first type of example has to do with the inexistence of demands that satisfy SARP. This case arises solely from the supports of the observed distributions of prices and its lack of rationality has nothing to do with the actual probabilities observed therein. I therefore refer to this case as "Inconsistent Supports." If this were the only kind of example, one could suspect that there is a stronger version of the results of section 5 which does not need to utilize any variables regarding the actual values of the probabilities and that, therefore, the testable restrictions we are obtaining are just a relaxation of the ones found by Brown and Matzkin, in which we allow for $\#\Omega$-many instances of SARP per individual instead of just one. The second type of example shows that this is not the case, as it shows that under consistent supports, there are values of the actual probabilities which are impossible to rationalize. I will refer to this case as "Inconsistent Probabilities." The existence of this example implies that the existence clauses for $g_{B,C}$, d_ω and $f_{\omega,e,p}$ in theorems 1 and 2 were not trivial.

6.1 Inconsistent supports:

Consider figure 5, where endowments $e, e' \in \left(\mathbb{R}^L_{++}\right)^2$ and prices $\overline{p}, \overline{p}' \in \mathcal{S}$ are illustrated.

Suppose that the supports of χ_e and $\chi_{e'}$ are given as in figure 6.

I now show that for no Ω can these data be Ω-rationalized.

I argue by contradiction. Suppose that for some Ω these data are rationalizable. Let $p \in \mathit{Supp}\,(\chi_e)$. By definition, there must exist $\omega \in \Omega$ such that $(\delta(\omega) > 0 \wedge p \in W_{u(\omega),e})$. Fix one such ω. Since $\delta(\omega) > 0$, there must exist $p' \in W_{u(\omega),e'}$ such that $p' \in \mathit{Supp}\,(\chi_{e'})$. But, then, consider figure 7, where I have drawn arbitrary $p \in \mathit{Supp}\,(\chi_e)$ and $p' \in \mathit{Supp}\,(\chi_{e'})$ and have highlighted the regions of consumer 1's budget that are feasible given the aggregate endowments and market clearing. Whatever $p \in \mathit{Supp}\,(\chi_e)$ and $p' \in \mathit{Supp}\,(\chi_{e'})$ are, it is impossible that consumer 1 satisfy the weak axiom of revealed preferences, and therefore it cannot be that $p \in W_{u(\omega),e}$ and $p' \in W_{u(\omega),e'}$.

Indeed, this example is extreme in that all the prices in $\mathit{Supp}\,(\chi_e)$ are inconsistent[15] with all the prices in $\mathit{Supp}\,(\chi_{e'})$. Clearly, it suffices that there exist one price in either one of the supports which is inconsistent with all the prices in the other support. Conversely, if a data set is such that for each price in the support of the price distribution at given endowments there exists at least one consistent price in the support of the price distribution at all other observed

[15] In the sense that, given market clearing, at least one consumer would have to violate the axioms of revealed preferences and, hence, individual rationality.

endowments, then there is no *a priori* reason to rule out rationalizability. The next example shows that in such a case the actual values of the probabilities matter, so that there is no *a priori* reason to imply rationalizability either.

6.2 Inconsistent probabilities:

Consider figure 8, which resembles the example of section 3. That is, suppose that endowments e and e' and associated distributions of prices χ_e and $\chi_{e'}$ have been observed such that $Supp(\chi_e) = Supp(\chi_{e'}) = \{\widehat{p}, \widetilde{p}\}$.

Figure 9 overlaps the previous two figures. It follows from either section 3 or the remarks at the end of the previous subsection that $Supp(\chi_e)$ and $Supp(\chi_{e'})$ are consistent. Nonetheless, the following claim establishes that not every arbitrary values of $\chi_{e'}(\{\widehat{p}\})$ and $\chi_e(\{\widetilde{p}\})$ can be rationalized for given set of events Ω. The claim is based on figure 9 and is to apply only for this example.

Claim 1 *For every set of events Ω, a data set $\{\{e, e'\}, (\chi_e, \chi_{e'})\}$ is Ω-rationalizable only if $\chi_{e'}(\{\widehat{p}\}) + \chi_e(\{\widetilde{p}\}) \geqslant 1$.*

Proof. Without loss of generality, suppose that $\{\{e, e'\}, (\chi_e, \chi_{e'})\}$ is Ω-rationalized by $u : \Omega \longrightarrow \mathcal{U}^2$, $\delta : \mathcal{P}(\Omega) \longrightarrow [0, 1]$ and $\varphi : u(\Omega) \times \{e, e'\} \longrightarrow F$ such that the support of δ is Ω.[16] Denote:

$$\begin{aligned}
\Omega_1 &= \left\{\omega \in \Omega | \, \widetilde{p} \in W_{u(\omega),e'} \wedge \widetilde{p} \in W_{u(\omega),e}\right\} \\
\Omega_2 &= \left\{\omega \in \Omega | \, \widehat{p} \in W_{u(\omega),e} \wedge \widehat{p} \in W_{u(\omega),e'}\right\} \qquad (58) \\
\Omega_3 &= \left\{\omega \in \Omega | \, \widehat{p} \in W_{u(\omega),e'} \wedge \widetilde{p} \in W_{u(\omega),e}\right\}
\end{aligned}$$

By SARP, $\Omega_1 \cap \Omega_2 = \varnothing$. Now, suppose that $\omega \in \Omega \backslash (\Omega_1 \cup \Omega_2)$. Then,

$$\neg \left(\left(\widetilde{p} \in W_{u(\omega),e'} \wedge \widetilde{p} \in W_{u(\omega),e} \right) \vee \left(\widehat{p} \in W_{u(\omega),e} \wedge \widehat{p} \in W_{u(\omega),e'} \right) \right) \qquad (59)$$

[16] That is that $\forall \omega \in \Omega$, $\delta(\{\omega\}) > 0$. If this is not initially the case, Ω can be trivially reduced so that the assumption holds.

which is

$$\neg\left(\widetilde{p}\in W_{u(\omega),e'}\wedge\widetilde{p}\in W_{u(\omega),e}\right)\wedge\neg\left(\widehat{p}\in W_{u(\omega),e}\wedge\widehat{p}\in W_{u(\omega),e'}\right) \tag{60}$$

or, equivalently,

$$\left(\widetilde{p}\notin W_{u(\omega),e'}\vee\widetilde{p}\notin W_{u(\omega),e}\right)\wedge\left(\widehat{p}\notin W_{u(\omega),e}\vee\widehat{p}\notin W_{u(\omega),e'}\right) \tag{61}$$

This implies that

$$\begin{aligned}&\left(\widetilde{p}\notin W_{u(\omega),e'}\wedge\widehat{p}\notin W_{u(\omega),e}\right)\\ &\vee\left(\widetilde{p}\notin W_{u(\omega),e'}\wedge\widehat{p}\notin W_{u(\omega),e'}\right)\\ &\vee\left(\widetilde{p}\notin W_{u(\omega),e}\wedge\widehat{p}\notin W_{u(\omega),e}\right)\\ &\vee\left(\widetilde{p}\notin W_{u(\omega),e}\wedge\widehat{p}\notin W_{u(\omega),e'}\right)\end{aligned} \tag{62}$$

and therefore, given the data set, that

$$\begin{aligned}&\left(\widehat{p}\in W_{u(\omega),e'}\wedge\widetilde{p}\in W_{u(\omega),e}\right)\\ &\vee\left(\widehat{p}\in W_{u(\omega),e'}\wedge\widehat{p}\notin W_{u(\omega),e'}\right)\\ &\vee\left(\widehat{p}\in W_{u(\omega),e}\wedge\widehat{p}\notin W_{u(\omega),e}\right)\\ &\vee\left(\widehat{p}\in W_{u(\omega),e}\wedge\widetilde{p}\in W_{u(\omega),e'}\right)\end{aligned} \tag{63}$$

The second and third possibilities are self-contradictory, whereas the fourth one is impossible by SARP. Hence, $\left(\widehat{p}\in W_{u(\omega),e'}\wedge\widetilde{p}\in W_{u(\omega),e}\right)$ and $\omega\in\Omega_3$. This proves that $\Omega_1\cup\Omega_2\cup\Omega_3=\Omega$.

Suppose now that $\omega\in\Omega_1\backslash\Omega_3$. Then,

$$\left(\widetilde{p}\in W_{u(\omega),e'}\wedge\widetilde{p}\in W_{u(\omega),e}\right)\wedge\neg\left(\widehat{p}\in W_{u(\omega),e'}\wedge\widetilde{p}\in W_{u(\omega),e}\right) \tag{64}$$

which implies that

$$\left(\widetilde{p} \in W_{u(\omega),e'} \wedge \widetilde{p} \in W_{u(\omega),e}\right) \wedge \left(\widehat{p} \notin W_{u(\omega),e'} \vee \widehat{p} \notin W_{u(\omega),e}\right) \tag{65}$$

and hence that

$$\widetilde{p} \in W_{u(\omega),e'} \wedge \widetilde{p} \in W_{u(\omega),e} \wedge \widehat{p} \notin W_{u(\omega),e'} \tag{66}$$

Moreover, by WARP, since $\widetilde{p} \in W_{u(\omega),e'}$, it follows that

$$\widetilde{p} \in W_{u(\omega),e'} \wedge \widetilde{p} \in W_{u(\omega),e} \wedge \widehat{p} \notin W_{u(\omega),e'} \wedge \widehat{p} \notin W_{u(\omega),e} \tag{67}$$

which implies that

$$\begin{aligned} \varphi\left(u\left(\omega\right), e'\right)\left(\{\widetilde{p}\}\right) &= 1 \\ \varphi\left(u\left(\omega\right), e'\right)\left(\{\widehat{p}\}\right) &= 0 \\ \varphi\left(u\left(\omega\right), e\right)\left(\{\widetilde{p}\}\right) &= 1 \\ \varphi\left(u\left(\omega\right), e\right)\left(\{\widehat{p}\}\right) &= 0 \end{aligned} \tag{68}$$

On the other hand, suppose that $\omega \in \Omega_2 \backslash \Omega_3$. Then,

$$\left(\widehat{p} \in W_{u(\omega),e} \wedge \widehat{p} \in W_{u(\omega),e'}\right) \wedge \neg\left(\widetilde{p} \in W_{u(\omega),e'} \wedge \widetilde{p} \in W_{u(\omega),e}\right) \tag{69}$$

from where

$$\left(\widehat{p} \in W_{u(\omega),e} \wedge \widehat{p} \in W_{u(\omega),e'}\right) \wedge \left(\widetilde{p} \notin W_{u(\omega),e'} \vee \widetilde{p} \notin W_{u(\omega),e}\right) \tag{70}$$

and

$$\widehat{p} \in W_{u(\omega),e} \wedge \widehat{p} \in W_{u(\omega),e'} \wedge \widetilde{p} \notin W_{u(\omega),e} \tag{71}$$

Again, by WARP, since $\widehat{p} \in W_{u(\omega),e}$, it follows that

$$\widehat{p} \in W_{u(\omega),e} \wedge \widehat{p} \in W_{u(\omega),e'} \wedge \widetilde{p} \notin W_{u(\omega),e} \wedge \widetilde{p} \notin W_{u(\omega),e'} \tag{72}$$

and hence that

$$\begin{aligned}
\varphi\left(u\left(\omega\right),e\right)\left(\{\widehat{p}\}\right) &= 1 \\
\varphi\left(u\left(\omega\right),e\right)\left(\{\widetilde{p}\}\right) &= 0 \\
\varphi\left(u\left(\omega\right),e'\right)\left(\{\widehat{p}\}\right) &= 1 \\
\varphi\left(u\left(\omega\right),e'\right)\left(\{\widetilde{p}\}\right) &= 0
\end{aligned} \tag{73}$$

Consider now the case when $\omega \in \Omega_3$. By definition, $\left(\widehat{p} \in W_{u(\omega),e'} \wedge \widetilde{p} \in W_{u(\omega),e}\right)$ whereas, by SARP $\neg\left(\widetilde{p} \in W_{u(\omega),e'} \wedge \widehat{p} \in W_{u(\omega),e}\right)$, which means that

$$\left(\widehat{p} \in W_{u(\omega),e'} \wedge \widetilde{p} \in W_{u(\omega),e}\right) \wedge \left(\widetilde{p} \notin W_{u(\omega),e'} \vee \widehat{p} \notin W_{u(\omega),e}\right) \tag{74}$$

Hence,

$$\begin{aligned}
&\left(\varphi\left(u\left(\omega\right),e'\right)\left(\{\widehat{p}\}\right) = 1 \wedge \varphi\left(u\left(\omega\right),e'\right)\left(\{\widetilde{p}\}\right) = 0\right) \\
&\vee \left(\varphi\left(u\left(\omega\right),e\right)\left(\{\widetilde{p}\}\right) = 1 \wedge \varphi\left(u\left(\omega\right),e\right)\left(\{\widehat{p}\}\right) = 0\right)
\end{aligned} \tag{75}$$

Now, by Ω-rationalizability, the previous results imply that

$$\begin{aligned}
\chi_{e'}\left(\{\widehat{p}\}\right) &= \sum_{\omega\in\Omega}\delta\left(\omega\right)\varphi\left(u\left(\omega\right),e'\right)\left(\{\widehat{p}\}\right) \\
&= \sum_{\omega\in\Omega_1\setminus\Omega_3}\delta\left(\omega\right)\varphi\left(u\left(\omega\right),e'\right)\left(\{\widehat{p}\}\right) \\
&\quad+\sum_{\omega\in\Omega_2\setminus\Omega_3}\delta\left(\omega\right)\varphi\left(u\left(\omega\right),e'\right)\left(\{\widehat{p}\}\right) \\
&\quad+\sum_{\omega\in\Omega_3}\delta\left(\omega\right)\varphi\left(u\left(\omega\right),e'\right)\left(\{\widehat{p}\}\right) \\
&= \sum_{\omega\in\Omega_2\setminus\Omega_3}\delta\left(\omega\right)+\sum_{\omega\in\Omega_3}\delta\left(\omega\right)\varphi\left(u\left(\omega\right),e'\right)\left(\{\widehat{p}\}\right)
\end{aligned} \tag{76}$$

whereas

$$\begin{aligned}
\chi_{e}\left(\{\widetilde{p}\}\right) &= \sum_{\omega\in\Omega}\delta\left(\omega\right)\varphi\left(u\left(\omega\right),e\right)\left(\{\widetilde{p}\}\right) \\
&= \sum_{\omega\in\Omega_1\setminus\Omega_3}\delta\left(\omega\right)\varphi\left(u\left(\omega\right),e\right)\left(\{\widetilde{p}\}\right) \\
&\quad+\sum_{\omega\in\Omega_2\setminus\Omega_3}\delta\left(\omega\right)\varphi\left(u\left(\omega\right),e\right)\left(\{\widetilde{p}\}\right) \\
&\quad+\sum_{\omega\in\Omega_3}\delta\left(\omega\right)\varphi\left(u\left(\omega\right),e\right)\left(\{\widetilde{p}\}\right) \\
&= \sum_{\omega\in\Omega_1\setminus\Omega_3}\delta\left(\omega\right)+\sum_{\omega\in\Omega_3}\delta\left(\omega\right)\varphi\left(u\left(\omega\right),e\right)\left(\{\widetilde{p}\}\right)
\end{aligned} \tag{77}$$

Then,

$$\begin{aligned}
\chi_{e'}\left(\{\widehat{p}\}\right)+\chi_{e}\left(\{\widetilde{p}\}\right) &= \sum_{\omega\in\Omega_2\setminus\Omega_3}\delta\left(\omega\right)+\sum_{\omega\in\Omega_1\setminus\Omega_3}\delta\left(\omega\right) \qquad (78)\\
&\quad+\sum_{\omega\in\Omega_3}\delta\left(\omega\right)\left(\varphi\left(u\left(\omega\right),e'\right)\left(\{\widehat{p}\}\right)+\varphi\left(u\left(\omega\right),e\right)\left(\{\widetilde{p}\}\right)\right) \\
&\geqslant \sum_{\omega\in\Omega_2\setminus\Omega_3}\delta\left(\omega\right)+\sum_{\omega\in\Omega_1\setminus\Omega_3}\delta\left(\omega\right)+\sum_{\omega\in\Omega_3}\delta\left(\omega\right) \\
&= 1
\end{aligned}$$

where the inequality comes from the fact that, as implied by previous results, $\forall \omega \in \Omega_3$,

$$\varphi\left(u\left(\omega\right), e'\right)\left(\{\widehat{p}\}\right) + \varphi\left(u\left(\omega\right), e\right)\left(\{\widetilde{p}\}\right) \geqslant 1 \tag{79}$$

given that

$$\varphi\left(u\left(\omega\right), e'\right)\left(\{\widehat{p}\}\right) = 1 \vee \varphi\left(u\left(\omega\right), e\right)\left(\{\widetilde{p}\}\right) = 1 \tag{80}$$

■

Hence, probabilities such that $\chi_e\left(\{\widetilde{p}\}\right) + \chi_{e'}\left(\{\widehat{p}\}\right) < 1$ are not rationalizable, in spite of the consistency of the supports.

7 Quantifier-free testable restrictions:

In this section I show that given the set of states of the world Ω, the observed set of profiles of endowments E and, for each observed profile of endowments $e \in E$, the set of observed prices $Supp\left(\chi_e\right)$, there exist restrictions (free of existential quantifiers) on the values of the probabilities that these observed prices can take. Moreover, I find, in abstract, the general functional form of these restrictions.

Before obtaining the result, a new and straightforward characterization of rationalizability is introduced. This characterization may appear notationally, and perhaps operationally, simpler that the one provided by the results of section 5. To my mind, however, this new characterization is much less interesting by itself, as it fails to uncover a fundamental feature of the theory of random preferences, namely the randomness of choices, which was fully done in that section. That is the reason why I have chosen to defer this new characterization until now.

Theorem 3 *A data set $\left\{E, \left(\chi_e\right)_{e\in E}\right\}$ is Ω-rationalizable if, and only if,:*

- *For each* $i \in \mathcal{I}$*, each* $B^i \in \mathcal{B}^i$ *and each* $\omega \in \Omega$*, there exist* $x^{i,B^i,\omega} \in \mathbb{R}_+^L$*,* $\lambda^{i,B^i,\omega} \in \mathbb{R}_{++}$ *and* $V^{i,B^i,\omega} \in \mathbb{R}$*;*

- *For each* $\omega \in \Omega$*, there exists* $d_\omega \in \mathbb{R}_+$*;*

- *For each* $\omega \in \Omega$*, each* $e \in E$ *and each* $p \in Supp\left(\chi_e\right)$*, there exists* $f_{\omega,e,p} \in \mathbb{R}_+$*;*

  such that:

1. *For each* $i \in \mathcal{I}$*, each* $\omega \in \Omega$*, each* $e, \widetilde{e} \in E$*, each* $p \in Supp\left(\chi_e\right)$ *and each* $\widetilde{p} \in Supp\left(\chi_{\widetilde{e}}\right)$*,*

$$V^{i,B\left(\widetilde{p},\widetilde{e}^i\right),\omega} \leqslant V^{i,B\left(p,e^i\right),\omega} + \lambda^{i,B\left(p,e^i\right),\omega} p \cdot \left(x^{i,B\left(\widetilde{p},\widetilde{e}^i\right),\omega} - x^{i,B\left(p,e^i\right),\omega}\right) \quad (81)$$

   with strict inequality if

$$x^{i,B\left(\widetilde{p},\widetilde{e}^i\right),\omega} \neq x^{i,B\left(p,e^i\right),\omega} \quad (82)$$

2. *For each* $i \in \mathcal{I}$*, each* $\omega \in \Omega$*, each* $e \in E$ *and each* $p \in Supp\left(\chi_e\right)$*,*

$$p \cdot x^{i,B\left(p,e^i\right),\omega} = p \cdot e^i \quad (83)$$

3.

$$\sum_{\omega \in \Omega} d_\omega = 1 \quad (84)$$

4. *For each* $\omega \in \Omega$*, each* $e \in E$ *and each* $p \in Supp\left(\chi_e\right)$*,*

$$\sum_{i \in \mathcal{I}} x^{i,B\left(p,e^i\right),\omega} \neq \sum_{i \in \mathcal{I}} e^i \Longrightarrow f_{\omega,e,p} = 0 \quad (85)$$

5. *For each $\omega \in \Omega$ and each $e \in E$,*

$$d_\omega > 0 \Longrightarrow \sum_{p \in Supp(\chi_e)} f_{\omega,e,p} = 1 \tag{86}$$

6. *For each $e \in E$ and each $p \in Supp\left(\chi_e\right)$,*

$$\chi_e\left(\{p\}\right) = \sum_{\omega \in \Omega} d_\omega f_{\omega,e,p} \tag{87}$$

Proof. Necessity: Suppose that $\delta : \mathcal{P}(\Omega) \longrightarrow [0,1]$, $u : \Omega \longrightarrow (\mathcal{U})^I$ and $\varphi : u[\Omega] \longrightarrow F$ Ω-rationalize $\left\{E, (\chi_e)_{e \in E}\right\}$. Denoting, $\forall \omega \in \Omega$, $u(\omega) = \left(u^i(\omega)\right)_{i \in \mathcal{I}}$, implicitly define $\forall i \in \mathcal{I}$, $\forall B^i \in \mathcal{B}^i$ and $\forall \omega \in \Omega$, $x^{i,B^i,\omega} \in \mathbb{R}_+^L$ by[17]

$$\left\{x^{i,B^i,\omega}\right\} = Arg \max_{y \in B^i} u^i(\omega)(y) \tag{88}$$

which one can do since each B^i is nonempty and compact and each $u^i(\omega)$ is continuous and strongly concave. Denote:

$$x^{i,B^i,\omega} = \arg\max_{y \in B^i} u^i(\omega)(y) \tag{89}$$

Then, $\forall i \in \mathcal{I}$ and $\forall \omega \in \Omega$, theorem 2 in Matzkin and Richter (1991), (b)$\Longrightarrow$(c) implies that $\forall i \in \mathcal{I}$, $\forall B^i \in \mathcal{B}^i$ and $\forall \omega \in \Omega$, $\exists \lambda^{i,B^i,\omega} \in \mathbb{R}_{++}$ and $\exists V^{i,B^i,\omega} \in \mathbb{R}$ that satisfy condition *1*, whereas condition *2* follows, by construction, from the monotonicity of $u^i(\omega)$.

Let $\forall \omega \in \Omega$, $d_\omega = \delta\left(\{\omega\}\right)$. The fact that $\forall \omega \in \Omega$, $d_\omega \in \mathbb{R}_+$ and condition *3* follow directly from the fact that δ is a probability measure.

Define $\forall \omega \in \Omega$, $\forall e \in E$ and $\forall p \in Supp\left(\chi_e\right)$, $f_{\omega,e,p} = \varphi\left(u(\omega), e\right)\left(\{p\}\right)$, and

[17] The notational proviso of note 8 applies here: $u^i(\omega)$ is a function mapping $\mathbb{R}_+^L$ into $\mathbb{R}$, which takes the value $u^i(\omega)(y)$ at $y \in \mathbb{R}_+^L$.

suppose that for $\omega \in \Omega$, $e \in E$ and $p \in Supp\left(\chi_e\right)$,

$$\sum_{i \in \mathcal{I}} x^{i, B\left(p, e^i\right), \omega} \neq \sum_{i \in \mathcal{I}} e^i \tag{90}$$

By construction, this means that

$$\sum_{i \in \mathcal{I}} \arg \max_{y \in B\left(p, e^i\right)} u^i\left(\omega\right)\left(y\right) \neq \sum_{i \in \mathcal{I}} e^i \tag{91}$$

which implies that $p \notin W_{u(\omega), e}$ and, then, since $\varphi\left(u\left(\omega\right), e\right)\left(W_{u(\omega), e}\right) = 1$, it follows that $\varphi\left(u\left(\omega\right), e\right)\left(\{p\}\right) = 0$ and therefore that $f_{\omega, e, p} = 0$, implying condition *4*.

Now, let $\widetilde{\omega} \in \Omega$ and $e \in E$ and suppose that $d_{\widetilde{\omega}} > 0$ and $\sum_{p \in Supp\left(\chi_e\right)} f_{\omega, e, p} \neq 1$. By construction, since $\varphi\left(u\left(\widetilde{\omega}\right), e\right) \in \digamma$, it must be that

$$\sum_{p \in Supp\left(\chi_e\right)} \varphi\left(u\left(\widetilde{\omega}\right), e\right)\left(\{p\}\right) < 1 \tag{92}$$

which implies that $\exists C \subseteq \mathcal{S} \backslash Supp\left(\chi_e\right) : \varphi\left(u\left(\widetilde{\omega}\right), e\right)\left(C\right) > 0$. Fix one such C. Since $\delta\left(\widetilde{\omega}\right) = d_{\widetilde{\omega}} > 0$, it follows that

$$\begin{aligned} \chi_e\left(C\right) &= \sum_{\omega \in \Omega} \delta\left(\omega\right) \varphi\left(u\left(\omega\right), e\right)\left(C\right) \\ &\geqslant \delta\left(\widetilde{\omega}\right) \varphi\left(u\left(\widetilde{\omega}\right), e\right)\left(C\right) \\ &> 0 \end{aligned} \tag{93}$$

contradicting the fact that $C \subseteq \mathcal{S} \backslash Supp\left(\chi_e\right)$. This implies condition *5*.

Finally, by construction, $\forall e \in E$ and $\forall p \in Supp(\{p\})$,

$$\begin{aligned} \chi_e(\{p\}) &= \sum_{\omega \in \Omega} \delta(\omega) \varphi(u(\omega), e)(\{p\}) \\ &= \sum_{\omega \in \Omega} d_\omega f_{\omega,e,p} \end{aligned} \tag{94}$$

which is condition *6*.

Sufficiency: By conditions *1* and *2* and theorem 2 in Matzkin and Richter (1991), (c)$\Longrightarrow$(b), it follows that $\forall i \in \mathcal{I}$, $\forall \omega \in \Omega$, $\exists U^{i,\omega} \in \mathcal{U}$ such that

$$(\forall e \in E)(\forall p \in Supp(\chi_e)) : x^{i,B\left(p,e^i\right),\omega} = \arg\max_{y \in B(p,e^i)} U^{i,\omega}(y) \tag{95}$$

Define the function $u : \Omega \longrightarrow \mathcal{U}^I$ by $(\forall \omega \in \Omega) : u(\omega) = \left(U^{i,\omega}\right)_{i \in \mathcal{I}}$.

Let $S = \#\Omega \in \mathbb{N}$ and denumerate $\Omega = \{\omega_1, \omega_2, ..., \omega_S\}$. Consider the following algorithm:

Algorithm 1 *Input:* Ω

1. $s = 1$, $\Theta = \varnothing$.
2. *If* $(\exists \widetilde{\omega} \in \Theta) : u(\widetilde{\omega}) = u(\omega)$, *then* $\theta = \varnothing$ *and go to 4.*
3. $\theta = \{\omega_s\}$
4. $\Theta = \Theta \cup \theta$
5. *If* $s = S$, *then* $\widetilde{\Omega} = \Theta$ *and stop.*
6. $s = s + 1$ *and go to 2.*

Output: $\widetilde{\Omega}$

The output of the algorithm, $\widetilde{\Omega} \subseteq \Omega$, has the properties that:

$$\left(\forall \omega, \widetilde{\omega} \in \widetilde{\Omega}\right) : \omega \neq \widetilde{\omega} \Longrightarrow u(\omega) \neq u(\widetilde{\omega}) \tag{96}$$

$$\left(\forall\omega\in\Omega\backslash\widetilde{\Omega}\right)\left(\exists\widetilde{\omega}\in\widetilde{\Omega}\right):u\left(\widetilde{\omega}\right)=u\left(\omega\right)\tag{97}$$

Define now the function $\delta:\mathcal{P}\left(\Omega\right)\longrightarrow\mathbb{R}$ as follows:

$$\begin{aligned}\left(\forall\omega\in\widetilde{\Omega}\right) &:& \delta\left(\left\{\omega\right\}\right)=d_{\omega}+\sum_{\widetilde{\omega}\in\Omega\backslash\widetilde{\Omega}:u(\widetilde{\omega})=u(\omega)}d_{\widetilde{\omega}}\\ \left(\forall\omega\in\Omega\backslash\widetilde{\Omega}\right) &:& \delta\left(\left\{\omega\right\}\right)=0 \\ \left(\forall\Phi\in\mathcal{P}\left(\Omega\right):\#\Phi\neq1\right) &:& \delta\left(\Phi\right)=\sum_{\omega\in\Phi}\delta\left(\left\{\omega\right\}\right)\end{aligned}\tag{98}$$

By condition *3* and construction, it follows that δ is a probability measure over Ω.

Now, define the function $\varphi:u\left[\Omega\right]\times E\longrightarrow F$ as follows. Fix $u\in u\left[\Omega\right]$ and $e\in E$. By definition and the second property of $\widetilde{\Omega}$, $\left\{\omega\in\Omega|\,u\left(\omega\right)=u\right\}\cap\widetilde{\Omega}\neq\varnothing$. Let $\omega_u\in\left\{\omega\in\Omega|\,u\left(\omega\right)=u\right\}\cap\widetilde{\Omega}$. By the first property of $\widetilde{\Omega}$, $\forall\widetilde{\omega}\in\widetilde{\Omega}\backslash\left\{\omega_u\right\}$, $u\left(\omega\right)\neq u$, from where

$$\#\left(\left\{\omega\in\Omega|\,u\left(\omega\right)=u\right\}\cap\widetilde{\Omega}\right)=1\tag{99}$$

and hence ω_u can be defined with no ambiguity. If $\delta\left(\left\{\omega_u\right\}\right)>0$, then define $\varphi\left(u,e\right):\Xi\longrightarrow\left[0,1\right]$ as:

$$\begin{aligned}\left(\forall p\in Supp\left(\chi_e\right)\right) &:& \varphi\left(u,e\right)\left(\left\{p\right\}\right)=\frac{d_{\omega_u}f_{\omega_u,e,p}+\sum_{\omega\in\Omega\backslash\widetilde{\Omega}:u(\omega)=u}d_{\omega}f_{\omega,e,p}}{\delta\left(\left\{\omega_u\right\}\right)}\\ \left(\forall p\in\mathcal{S}\backslash Supp\left(\chi_e\right)\right) &:& \varphi\left(u,e\right)\left(\left\{p\right\}\right)=0 \\ \left(\forall C\in\Xi:\#C\neq1\right) &:& \varphi\left(u,e\right)\left(C\right)=\sum_{p\in C}\varphi\left(u,e\right)\left(\left\{p\right\}\right)\end{aligned}\tag{100}$$

Notice that, by construction,

$$
\begin{aligned}
& \varphi(u,e)(\mathcal{S}) \\
= \; & \sum_{p\in\mathcal{S}} \varphi(u,e)(\{p\}) \\
= \; & \sum_{p\in Supp(\chi_e)} \varphi(u,e)(\{p\}) \\
= \; & \sum_{p\in Supp(\chi_e)} \frac{d_{\omega_u} f_{\omega_u,e,p} + \sum_{\omega\in\Omega\setminus\tilde{\Omega}:u(\omega)=u} d_\omega f_{\omega,e,p}}{\delta(\{\omega_u\})} \qquad (101) \\
= \; & \frac{d_{\omega_u} \sum_{p\in Supp(\chi_e)} f_{\omega_u,e,p} + \sum_{\omega\in\Omega\setminus\tilde{\Omega}:u(\omega)=u} \left(d_\omega \sum_{p\in Supp(\chi_e)} f_{\omega,e,p}\right)}{\delta(\{\omega_u\})} \\
= \; & \frac{d_{\omega_u} + \sum_{\omega\in\Omega\setminus\tilde{\Omega}:u(\omega)=u} d_\omega}{\delta(\{\omega_u\})} \\
= \; & 1
\end{aligned}
$$

where the fifth equality follows from condition *5*. This and the construction imply that $\varphi(u,e) \in \mathcal{F}$. If, alternatively, $\delta(\{p\}) = 0$, then let $p \in W_{u,e}$, which exists by Arrow and Debreu (1954), and define $\varphi(u,e) : \Xi \longrightarrow [0,1]$ by

$$
(\forall C \in \Xi) : \varphi(u,e)(C) = \begin{cases} 1 & \text{if } \; p \in C \\ 0 & \text{otherwise} \end{cases} \qquad (102)
$$

from where it is obvious that $\varphi(u,e) \in \mathcal{F}$.

I now show that the functions u, δ and φ Ω-rationalize the data set $\left\{E, (\chi_e)_{e\in E}\right\}$.

First, let $e \in E$ and $C \in \Xi$. Then, by construction,

$$\begin{aligned}
& \sum_{\omega\in\Omega} \delta(\{\omega\})\,\varphi(u(\omega),e)(C) \\
= & \sum_{\omega\in\Omega:\delta(\{\omega\})>0} \delta(\{\omega\})\,\varphi(u(\omega),e)(C) \\
= & \sum_{\omega\in\widetilde{\Omega}:\delta(\{\omega\})>0} \left(\delta(\{\omega\})\sum_{p\in C}\varphi(u(\omega),e)(\{p\})\right) \\
= & \sum_{\omega\in\widetilde{\Omega}:\delta(\{\omega\})>0} \sum_{p\in C}\delta(\{\omega\})\,\varphi(u(\omega),e)(\{p\}) \\
= & \sum_{\omega\in\widetilde{\Omega}:\delta(\{\omega\})>0} \sum_{p\in C}\left(d_\omega f_{\omega,e,p} + \sum_{\widetilde{\omega}\in\Omega\setminus\widetilde{\Omega}:u(\widetilde{\omega})=u(\omega)} d_{\widetilde{\omega}} f_{\widetilde{\omega},e,p}\right) \qquad (103) \\
= & \sum_{p\in C}\sum_{\omega\in\Omega} d_\omega f_{\omega,e,p} \\
= & \sum_{p\in C}\chi_e(\{p\}) \\
= & \chi_e(C)
\end{aligned}$$

where the fifth step follows from the properties of $\widetilde{\Omega}$ and the fact that $\forall\omega\in\widetilde{\Omega}$, $\delta(\{\omega\}) = 0$ implies that $d_\omega = 0$, and the previous to last step follows from property *6*.

Now, fix $\omega\in\Omega$ and $e\in E$. Suppose that for some $p\in\mathcal{S}$, $\varphi(u(\omega),e)(\{p\}) > 0$. By the second property of $\widetilde{\Omega}$, $\exists\widetilde{\omega}\in\widetilde{\Omega}$ such that $u(\widetilde{\omega}) = u(\omega)$. If $\delta(\widetilde{\omega}) = 0$, it follows by construction that

$$\begin{aligned}
p &\in W_{u(\widetilde{\omega}),e} \qquad (104) \\
&= W_{u(\omega),e}
\end{aligned}$$

If, on the other hand, $\delta(\widetilde{\omega}) > 0$, then, by construction,

$$(\exists\widehat{\omega}\in\Omega) : u(\widehat{\omega}) = u(\widetilde{\omega}) \wedge d_{\widehat{\omega}} > 0 \wedge f_{\widehat{\omega},e,p} > 0 \qquad (105)$$

By condition *4*,

$$\sum_{i\in\mathcal{I}} x^{i,B\left(p,e^{i}\right),\widehat{\omega}} = \sum_{i\in\mathcal{I}} e^{i} \tag{106}$$

and, hence, by construction,

$$\begin{aligned} p &\in W_{u(\widehat{\omega}),e} \\ &= W_{u(\widetilde{\omega}),e} \\ &= W_{u(\omega),e} \end{aligned} \tag{107}$$

This implies that $\varphi\left(u\left(\omega\right),e\right)\left(\mathcal{S}\backslash W_{u(\omega),e}\right)=0$, or that $\varphi\left(u\left(\omega\right),e\right)\left(W_{u(\omega),e}\right)=1$. ■

The previous characterization and appendix 9 allow for the following theorem, for which I need the following notation: define the functions $sgn:\mathbb{R}\longrightarrow\{-1,0,1\}$ by

$$sgn\left(x\right)=\begin{cases} -1 & \text{if} \quad x<0 \\ 0 & \text{if} \quad x=0 \\ 1 & \text{if} \quad x>0 \end{cases} \tag{108}$$

and $\overrightarrow{sgn}:\mathbb{R}^{L}\longrightarrow\{-1,0,1\}^{L}$ by

$$\overrightarrow{sgn}\left(x\right)=\left(sgn\left(x_{l}\right)\right)_{l=1}^{L} \tag{109}$$

Theorem 4 *Given Ω, let $E\in\left(\mathbb{R}_{+}^{L}\right)^{I}$ and for each $e\in E$, $Supp\left(\chi_{e}\right)\subseteq\mathcal{S}$ be* *given. Let* Ψ *be the set of vectors*[18]

$$\left(\left(\chi_{e,p}\right)_{p\in Supp(\chi_e)}\right)_{e\in E}\in\prod_{e\in E}[0,1]^{\#Supp(\chi_e)}$$

such that the data set

$$\left\{E,\left(C\longmapsto\sum_{p\in C}\chi_{e,p}\right)_{e\in E}\right\}\tag{110}$$

is Ω*-rationalizable.*[19] Ψ *is a semialgebraic set.*

Proof. It follows from theorem 3 that

$$\left(\left(\chi_{e,p}\right)_{p\in Supp(\chi_e)}\right)_{e\in E}\in\Psi\subseteq\prod_{e\in E}[0,1]^{\#Supp(\chi_e)}\tag{111}$$

if, and only, if there exists a vector

$$\zeta=\begin{pmatrix}\left(\left(\left(\left(x^{i,B\left(p,e^i\right),\omega}\right)_{p\in Supp(\chi_e)}\right)_{e\in E}\right)_{\omega\in\Omega}\right)_{i\in\mathcal{I}}\\ \left(\left(\left(\left(\lambda^{i,B\left(p,e^i\right),\omega}\right)_{p\in Supp(\chi_e)}\right)_{e\in E}\right)_{\omega\in\Omega}\right)_{i\in\mathcal{I}}\\ \left(\left(\left(\left(V^{i,B\left(p,e^i\right),\omega}\right)_{p\in Supp(\chi_e)}\right)_{e\in E}\right)_{\omega\in\Omega}\right)_{i\in\mathcal{I}}\\ \left(\left(\left(f_{\omega,e,p}\right)_{p\in Supp(\chi_e)}\right)_{e\in E}\right)_{\omega\in\Omega}\\ \left(d_\omega\right)_{\omega\in\Omega}\end{pmatrix}\tag{112}$$

[18] By a vector

$$\left(\left(z_{a,b}\right)_{b\in\{1,\dots,B\}}\right)_{a\in\{1,\dots,A\}}$$

I denote

$$\left(z_{1,1},z_{1,2},...,z_{1,B},z_{2,1},z_{2,2},...,z_{2,B},...,z_{A,1},z_{A,2},...,z_{A,B}\right)$$

[19] The notation $C\longmapsto\sum_{p\in C}\chi_{e,p}$ means that the function $\chi_e:\Xi\longrightarrow[0,1]$ is constructed as:

$$(\forall C\in\Xi):\chi_e(C)=\sum_{p\in C}\chi_{e,p}$$

in the Cartesian product of the sets

$$\left(\left(\prod_{e\in E}\left(\mathbb{R}_{+}^{L}\right)^{\#Supp(\chi_e)}\right)^{\#\Omega}\right)^{I} \tag{113}$$

$$\left(\left(\prod_{e\in E}\left(\mathbb{R}_{++}\right)^{\#Supp(\chi_e)}\right)^{\#\Omega}\right)^{I} \tag{114}$$

$$\left(\left(\prod_{e\in E}\left(\mathbb{R}\right)^{\#Supp(\chi_e)}\right)^{\#\Omega}\right)^{I} \tag{115}$$

$$\left(\prod_{e\in E}[0,1]^{\#Supp(\chi_e)}\right)^{\#\Omega} \tag{116}$$

$$[0,1]^{\#\Omega} \tag{117}$$

(which is a finite-dimensional Euclidean space), such that

$$\left(\left(\left(\chi_{e,p}\right)_{p\in Supp(\chi_e)}\right)_{e\in E},\zeta\right) \tag{118}$$

satisfies the following conditions:

(i) $\forall i\in\mathcal{I}$, $\forall\omega\in\Omega$, $\forall e,\widetilde{e}\in E$, $\forall p\in Supp\left(\chi_e\right)$ and $\forall\widetilde{p}\in Supp\left(\chi_{\widetilde{e}}\right)$,

$$\left(\begin{array}{c} sgn\left(V^{i,B\left(\widetilde{p},\widetilde{e}^{i}\right),\omega}-V^{i,B\left(p,e^{i}\right),\omega}\right. \\ \left.-\lambda^{i,B\left(p,e^{i}\right),\omega}p\cdot\left(x^{i,B\left(\widetilde{p},\widetilde{e}^{i}\right),\omega}-x^{i,B\left(p,e^{i}\right),\omega}\right)\right)=-1 \\ \vee \\ \left(\begin{array}{c} sgn\left(V^{i,B\left(\widetilde{p},\widetilde{e}^{i}\right),\omega}-V^{i,B\left(p,e^{i}\right),\omega}\right. \\ \left.-\lambda^{i,B\left(p,e^{i}\right),\omega}p\cdot\left(x^{i,B\left(\widetilde{p},\widetilde{e}^{i}\right),\omega}-x^{i,B\left(p,e^{i}\right),\omega}\right)\right)=0 \\ \wedge \\ \overrightarrow{sgn}\left(x^{i,B\left(\widetilde{p},\widetilde{e}^{i}\right),\omega}-x^{i,B\left(p,e^{i}\right),\omega}\right)=\left(0\right)_{l=1}^{L} \end{array}\right) \end{array}\right) \tag{119}$$

(ii) $\forall i \in \mathcal{I}$, $\forall \omega \in \Omega$, $\forall e \in E$ and $\forall p \in Supp\,(\chi_e)$, $sgn\left(p \cdot e^i - p \cdot x^{i,B\left(p,e^i\right),\omega}\right) = 0$.

(iii) $sgn\left(\sum_{\omega \in \Omega} d_\omega - 1\right) = 0$.

(iv) $\forall \omega \in \Omega$, $\forall e \in E$ and $\forall p \in Supp\,(\chi_e)$,

$$\overrightarrow{sgn}\left(f_{\omega,e,p}\left(\sum_{i \in \mathcal{I}} e_i - \sum_{i \in \mathcal{I}} x^{i,B\left(p,e^i\right),\omega}\right)\right) = (0)_{l=1}^{L} \tag{120}$$

(v) $\forall \omega \in \Omega$ and $\forall e \in E$,

$$sgn\left(d_\omega\left(\sum_{p \in Supp(\chi_e)} f_{\omega,e,p} - 1\right)\right) = 0 \tag{121}$$

(vi) $\forall e \in E$ and $\forall p \in Supp\,(\chi_e)$,

$$sgn\left(\chi_{e,p} - \sum_{\omega \in \Omega} d_\omega f_{\omega,e,p}\right) = 0 \tag{122}$$

Consider the set of vectors

$$\left(\left(\left(\chi_{e,p}\right)_{p \in Supp(\chi_e)}\right)_{e \in E}, \zeta\right) \tag{123}$$

that satisfy conditions (i) to (vi). By definition 4, such set is semialgebraic. By corollary 1, the projection of this set into $\prod_{e \in E} \mathbb{R}^{\#Supp(\chi_e)}$, which is precisely Ψ, is also semialgebraic. ■

It follows from Arrow and Debreu (1954) that the set Ψ introduced in the previous theorem need not be empty. The first example of section 6 shows that such set may be empty and, more interestingly, the second example in that section shows that, when nonempty, the set Ψ may be a proper subset of

$$\prod_{e \in E} [0,1]^{\#Supp(\chi_e)} \tag{124}$$

Then, there do exist testable restrictions on the set Ψ only, and these restrictions take, in abstract, the form of polynomial inequalities (although they could be of the form $sgn(1) = 0$, as shown by the first example of nonrationalizability).

8 Concluding remarks:

The goal of this paper has been to argue that general equilibrium theory is refutable, even without observation of individual choices and allowing individual preferences to vary randomly. This result goes in line with the ones of Brown and Matzkin (1996), which showed that the common belief that general equilibrium theory was unfalsifiable, as seemingly implied by the Sonnenschein-Mantel-Debreu literature of the Seventies, was overly pessimistic. My results, however, try to overcome the criticism, common in mathematical psychology, of the assumption of invariant preferences which is implicitly present in the work of Brown and Matzkin via their application of revealed-preference theory.

I have found that for a given finite economy, if one observes a finite set of profiles of individual endowments and, for each one of these profiles, a probability distribution of prices with finite support is also observed, there exists an exhaustive set of necessary conditions that have to be satisfied for the data to be consistent with general equilibrium theory, given a set of possible states of the world and allowing for random determination of individual preferences in these states of the world. These restrictions were studied here in two instances. Firstly, a characterization of the condition of consistency of data and theory was given via the existence of individual contingent demands and of probabilistic distributions of choices and equilibrium prices. Secondly, it was argued that these existential quantifiers can be eliminated, and that the conditions of the first characterization have an equivalent in terms of conditions purely on the data. These latter conditions were not explicitly obtained, and only their abstract mathematical form could be determined. However, I have also shown that they are not vacuous: they constitute a test of the consistency of data and general equilibrium theory with power to refute this hypothesis.

In the paper I have assumed that as the state of the world changes individuals realize that their preferences change and choose accordingly. This accommodates an interpretation of the theory of random choice found in the literature. An alternative interpretation is that individuals, although endowed with one preference relation, are unclear about their preferences and act accordingly to their perceptions of these preferences, which depend on the state of the world. This interpretation can be easily accommodated by my results. However, in both interpretations, if there are additional hypotheses about how different states of the world affect individual preferences, they need to be incorporated in the theory, since in my results I allow a very general class of preferences to be assigned to the individuals at each one of the different states of the world. The conditions found here should continue to be necessary, but my arguments for sufficiency may not accommodate these additional hypotheses, and hence the list of restrictions given here is no longer exhaustive.

9 Appendix: Tarski-Seidenberg quantifier elimination.

Some of the logical statements in the paper contain existential quantifiers on unobserved (and even unobservable) variables of their models. It is convenient to argue that these quantifiers can be eliminated and to obtain as much information as possible regarding equivalent statements that are free of quantifiers. For this, I used the classical theory of quantifier elimination presented here. This appendix takes concepts from Mishra (1993).

Definition 2 *A function $\mu : \mathbb{R}^K \longrightarrow \mathbb{R}$, where $K \in \mathbb{N}$, is a (Real) Multivariate Monomial if there exists $\{\alpha_k\}_{k=1}^{K} \overset{\text{seq}}{\subseteq} \mathbb{N} \cup \{0\}$ such that for every $x \in \mathbb{R}^K$,*

$$\mu(x) = \prod_{k=1}^{K} x_i^{\alpha_k} \tag{125}$$

The degree of the monomial is $\deg(\mu) = \sum_{k=1}^{K} \alpha_k$.

Definition 3 *A function $\rho : \mathbb{R}^K \longrightarrow \mathbb{R}$, where $K \in \mathbb{N}$, is a (Real) Multivariate Polynomial if for some $M \in \mathbb{N}$, there exist Multivariate Monomials $\left\{\mu_m : \mathbb{R}^K \longrightarrow \mathbb{R}\right\}_{m=1}^{M}$ and $\{a_m\}_{m=1}^{M} \overset{seq}{\subseteq} \mathbb{R} \backslash \{0\}$ such that,*

$$\rho = \sum_{m=1}^{M} a_m \mu_m \tag{126}$$

The degree of the polynomial is $\deg(\rho) = \max_{m \in \{1,\ldots,M\}} \{\deg(\mu_m)\}$.

Definition 4 *A set $A \subseteq \mathbb{R}^K$, where $K \in \mathbb{N}$, is a semialgebraic set if it can be determined by a set theoretic expression of the form*

$$A = \bigcup_{m=1}^{M} \bigcap_{n=1}^{N_m} \left\{ x \in \mathbb{R}^K \middle| \, sgn\left(\rho_{m,n}(x)\right) = s_{m,n} \right\} \tag{127}$$

where for each $m \in \{1, ..., M\}$, $M \in \mathbb{N}$ and each $n \in \{1, ..., N_m\}$, $N_m \in \mathbb{N}$, $\rho_{m,n} : \mathbb{R}^K \longrightarrow \mathbb{R}$ is a Multivariate Polynomial and $s_{n,m} \in \{-1, 0, 1\}$.

Definition 5 *A function $\eta : A \longrightarrow B$, where $A \subseteq \mathbb{R}^{K_A}$ and $B \subseteq \mathbb{R}^{K_B}$ are semialgebraic sets ($K_A, K_B \in \mathbb{N}$), is a semialgebraic map if its graph,*

$$Graph(\eta) = \left\{ (x, y) \in \mathbb{R}^{K_A} \times \mathbb{R}^{K_B} \middle| \, y = \eta(x) \right\} \tag{128}$$

is semialgebraic.

Theorem 5 *(The Tarski-Seidenberg Theorem:) Let $A \subseteq \mathbb{R}^K$, where $K \in \mathbb{N}$, be* *a semialgebraic set and let* $\eta : \mathbb{R}^K \longrightarrow \mathbb{R}^{K'}$*, where* $K' \in \mathbb{N}$*, be a semialgebraic map. Then,*

$$\eta[A] = \left\{ y \in \mathbb{R}^{K'} \middle| (\exists x \in A) : \eta(x) = y \right\} \tag{129}$$

is a semialgebraic set.

Proof. This is theorem 8.6.6 in Mishra (1993), pp. 345. ■

Corollary 1 *Let* $A \subseteq \mathbb{R}^{K_1} \times \mathbb{R}^{K_2}$*, where* $K_1, K_2 \in \mathbb{N}$*, be a semialgebraic set and let* $\overrightarrow{A}^1$ *be its projection into* $\mathbb{R}^{K_1}$*, defined as*

$$\overrightarrow{A}^1 = \left\{ x \in \mathbb{R}^{K_1} \middle| \left(\exists y \in \mathbb{R}^{K_2}\right) : (x, y) \in A \right\} \tag{130}$$

Then, $\overrightarrow{A}^1$ *is semialgebraic.*

Proof. Define the function $\eta_1 : \mathbb{R}^{K_1} \times \mathbb{R}^{K_2} \longrightarrow \mathbb{R}^{K_1}$ by $\eta_1(x, y) = x$. Its graph, $G(\eta_1) = \left(\mathbb{R}^{K_1} \times \mathbb{R}^{K_2}\right) \times \mathbb{R}^{K_1}$ is clearly semialgebraic. Since A is semialgebraic, it follows from the Tarski-Seidenberg theorem that

$$\begin{aligned} & \left\{ x \in \mathbb{R}^{K_1} \middle| (\exists (x', y) \in A) : \eta_1(x', y) = x \right\} \\ = & \left\{ x \in \mathbb{R}^{K_1} \middle| (\exists (x', y) \in A) : x' = x \right\} \\ = & \left\{ x \in \mathbb{R}^{K_1} \middle| \left(\exists y \in \mathbb{R}^{K_2}\right) : (x, y) \in A \right\} \\ = & \overrightarrow{A}^1 \end{aligned} \tag{131}$$

is semialgebraic. ■

10 References:

Allen, B., 1985, Continuous random selections from the equilibrium price correspondence, Working Paper No. 85-03, CARESS, University of Pennsylvania.

Arrow, K. and G. Debreu, 1954, Existence of an equilibrium for a competitive economy, Econometrica 22, 265-292.

Block, H.D. and J. Marshack, 1960, Random orderings and stochastic theories of responses, in I. Olkin, S. Ghurye, W. Hoeffding, W. Madow and H. Mann, eds., Contributions to Probability and Statistics, (Stanford University Press, Palo Alto) 97-132.

Brown, D. and R. Matzkin, 1996, Testable restrictions on the equilibrium manifold, Econometrica 64, 1249-1262.

Carvajal, A., 2002a, Testable restrictions of general equilibrium theory in exchange economies with externalities, manuscript, Brown University.

Carvajal, A., 2002b, Individually rational collective choice under random preferences, manuscript, Brown University.

Chiappori, P.A., I. Ekeland, F. Kubler and H. Polemarchakis, 2000, The identification of preferences from equilibrium prices, Discussion paper No. 2000/24, CORE, Université Catholique de Louvain.

Chiappori , P.A. and J.C. Rochet, 1987, Revealed preferences and differentiable demand, Econometrica 55, 687-691.

Chiappori, P.A. and I. Ekeland, 1999, Aggregation and market demand: an exterior differential calculus viewpoint, Econometrica 68, 1529-1540.

Chiappori, P.A., I. Ekeland, F. Kubler and H. Polemarchakis, 2002, Testable implications of general equilibrium theory: a differentiable approach, Working Paper 02-10, Department of Economics, Brown.

Constantinides, G. and D. Duffie, 1996, Asset pricing with heterogeneous consumers, Journal of Political Economy, 104, 219-240.

Debreu, G., 1970, Economies with a finite set of equilibria, Econometrica 38, 387-392.

Debreu, G., 1972, Smooth preferences, Econometrica 40, 603-615.

Debreu, G., 1974, Excess demand functions, Journal of Mathematical Economics 1, 15-23.

Diewert, W., 1977, Generalized Slutsky conditions for aggregate consumer demand functions, Journal of Economic Theory 7, 353-362.

Geanakoplos, J. and H. Polemarchakis, 1980, On the disaggregation of excess demand functions, Econometrica 48, 315-331.

Hausman, D., 2000, Revealed preference, belief, and game theory, Economics and Philosophy 16, 99-115.

Krebs, T., 2001, Testable implications of consumption-based asset pricing models with incomplete markets, manuscript, Brown University.

Kroeber-Riel, W., 1971, Constructs and empirical basis in theories of economic behavior, Theory and Decision 1, 337-349.

Kubler, F., 2001, Observable restrictions of general equilibrium with financial markets, manuscript, Stanford University.

Luce, R.D., 1959, Individual choice and behavior (John Wiley and Sons, New York).

Luce, R.D. and P. Supes, 1965, Preference, utility and subjective probability, in R.D. Luce, R. Bush and E. Gallanter, eds., Handbook of Mathematical Psychology, Vol III (John Wiley and Sons, New York) 249-40.

Mantel, R., 1974, On the characterization of aggregate excess demand, Journal of Economic Theory 7, 348-353.

Mantel, R., 1977, Implications of microeconomic theory for community excess demand functions, in M. Intriligator, ed., Frontiers of Quantitative Economics, Vol 3A (North-Holland, Amsterdam) 111-126.

Mas-Colell, A., 1977, On the equilibrium price set of an exchange economy, Journal of Mathematical Economics 4, 117-126

Matzkin, R. and M. Richter, 1991, Testing strictly concave rationality, Journal of Economic Theory 58, 287-303.

McFadden, D. and M. Richter, 1990, Stochastic rationality and stochastic revealed preference, in J. Chipman, D. McFadden and M. Richter, eds., Preferences, uncertainty and optimality (Westview Press, Boulder) 161-186.

Mishra, B., 1993, Algorithmic Algebra (Springer-Verlag, New York).

Shafer, W and H. Sonnenschein, 1982, Market demand and excess demand functions, in K. Arrow and M. Intriligator, eds., Handbook of Mathematical Economics (North-Holland, Amsterdam), 671-693.

Sonnenschein, H., 1973, Do Walras' identity and continuity characterize the class of community excess demand functions?, Journal of Economic Theory 6, 345-354.

Snyder, S., 1999, Testable restrictions of Pareto optimal public good provision, Journal of Public Economics 71, 97-199.

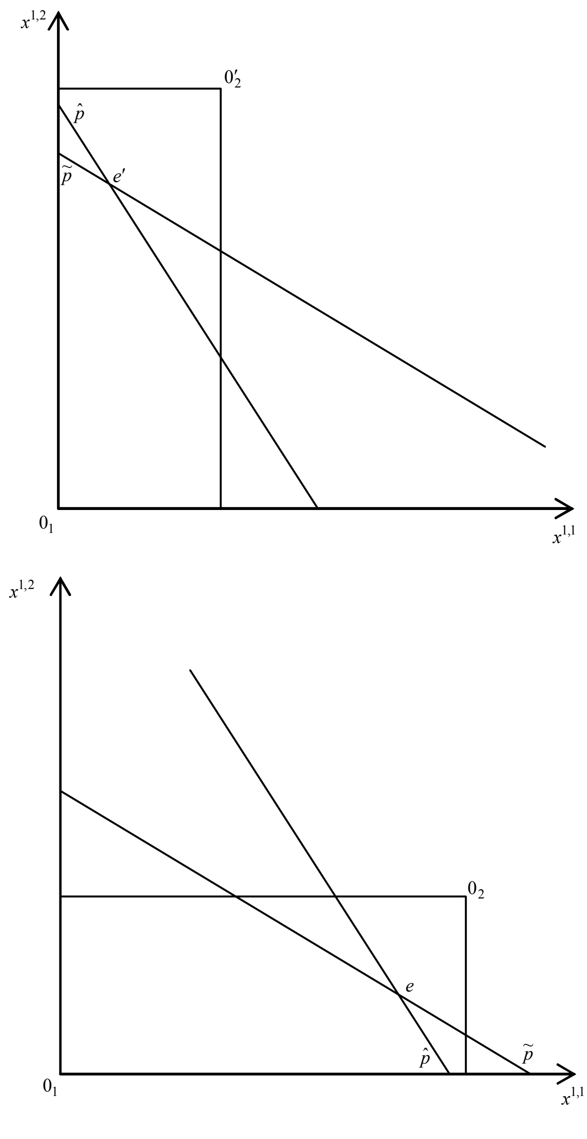


Figure 1:

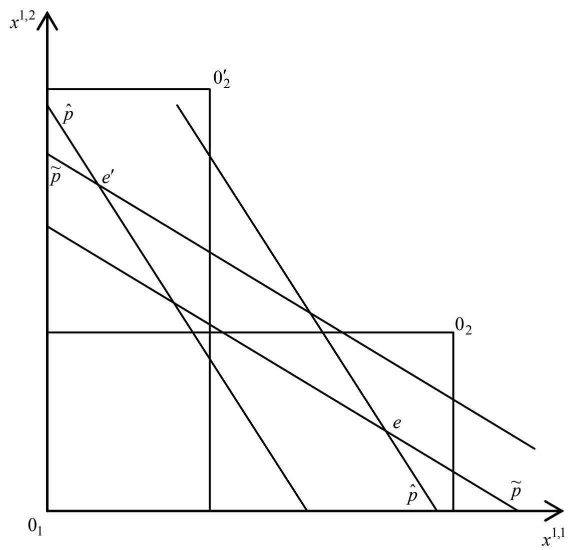


Figure 2:

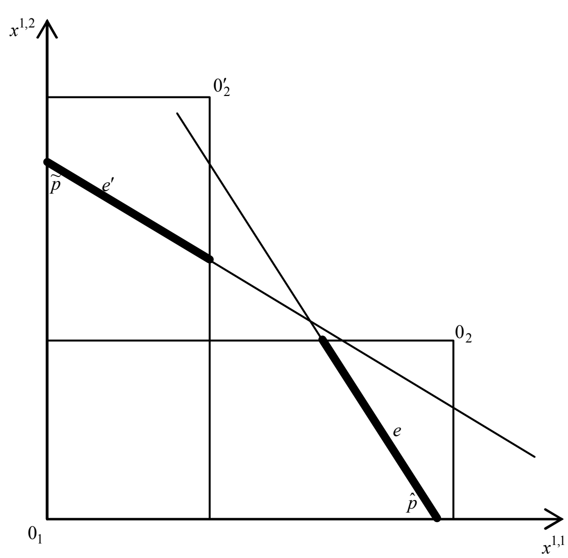
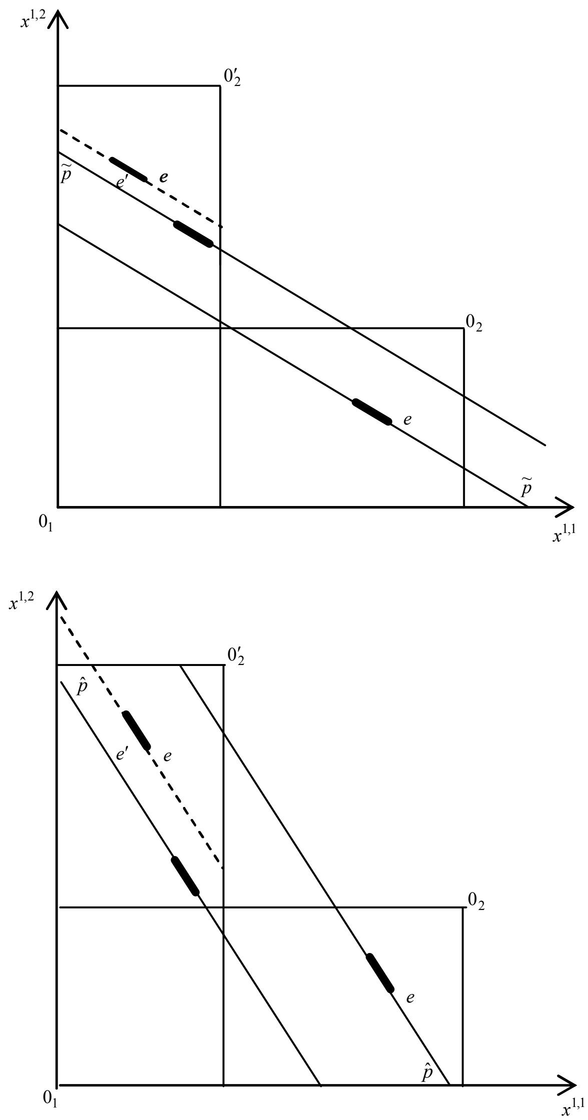


Figure 3:



$x^{1,2}$
$0_2'$
$\tilde{p}$
e'
e
0_2
e
$\tilde{p}$
0_1
$x^{1,1}$
$x^{1,2}$
$0_2'$
$\hat{p}$
e'
e
0_2
e
$\hat{p}$
0_1
$x^{1,1}$

Figure 4:

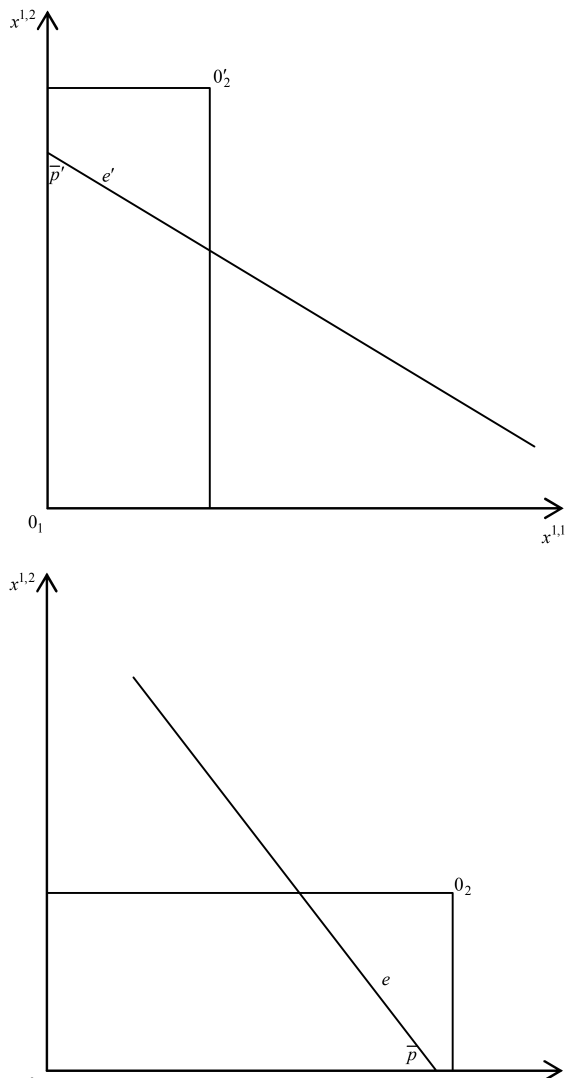


Figure 5:

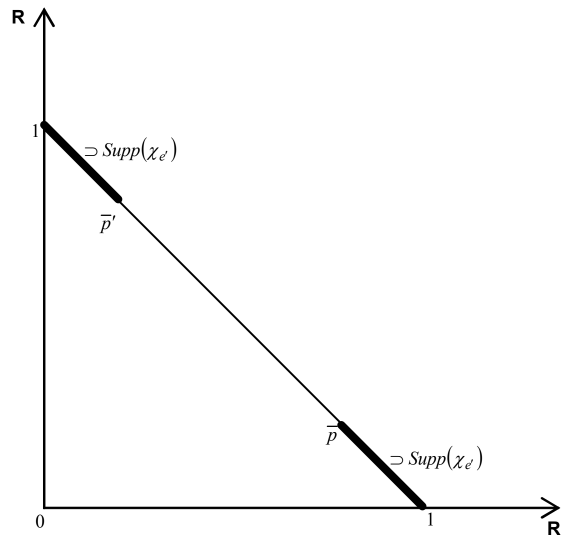


Figure 6:

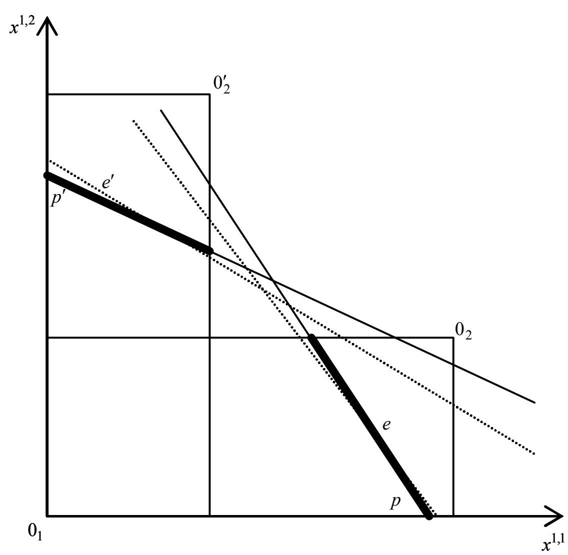


Figure 7:

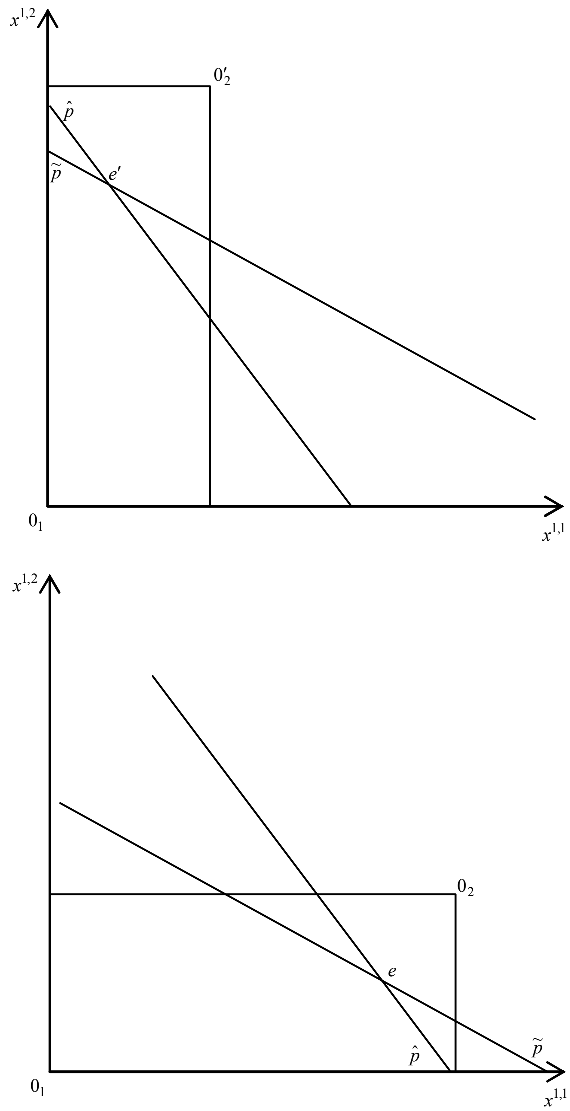


Figure 8:

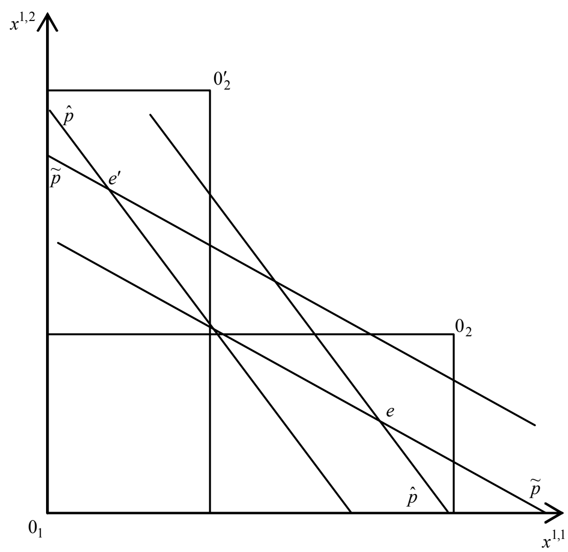


Figure 9: