# Asset value and securitization under heterogeneous beliefs

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November 26, 2014

# ASSET VALUE AND SECURITIZATION UNDER HETEROGENEOUS BELIEFS[1]

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JOB MARKET PAPER

This paper studies the effect of securitization on asset pricing when agents have heterogeneous beliefs about the stochastic process on dividends, prices and interest rates. For this purpose, the asset pricing model of Harrison and Kreps (1978) is modified to account for the possibility for agents to issue asset backed securities. The securities are constrained to belong to tranches of different payment priority, mimicking collateralized debt obligations (CDO). Securitization weakly increases the gap between the price of an underlying asset and any perceived present value of its dividends. A necessary condition for this increase to be strict is the absence of beliefs regarding the next-period price of the underlying asset which first-order stochastically dominate all other beliefs. In states of the world where investors with divergent beliefs buy securities from different tranches, the underlying asset is traded at a price higher than what anyone thinks it is worth. Since securities with a return below the market interest rate may be traded across agents, securitization has mixed effects on portfolio returns. In cases where there is a type of agent more sophisticated than all others, securitization can weakly decrease the returns all agents receive.

Keywords: asset pricing, securitization, tranching, belief disagreement.

## 1. INTRODUCTION

Securitization appears to be at the core of the financial crisis of 2007-2008 (Brunnermeier (2009)). Subprime mortgages were pooled together and sold in the form of mortgage backed securities (MBS). The cash flow from the mortgages was supposed to go to MBS holders, giving priority to MBS in senior tranches. Given this, senior tranches were seen as safe and were highly rated by rating agencies. Furthermore, even the MBS in junior tranches could be pooled with many others and resold in the form of collateralized debt

[1]Thanks to Jason Donaldson, Erik Eyster, Michele Piccione, Francesco Nava, Daniel Osorio, Balazs Szentes and attendants to the LSE Summer Seminar for comments and suggestions.

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obligations (CDO). Thanks to tranching, CDO in senior tranches were also highly rated (Figure 1). The possibility of creating high-rated securities from subprime mortgages eased the credit towards subprime borrowers, fueling a housing bubble.

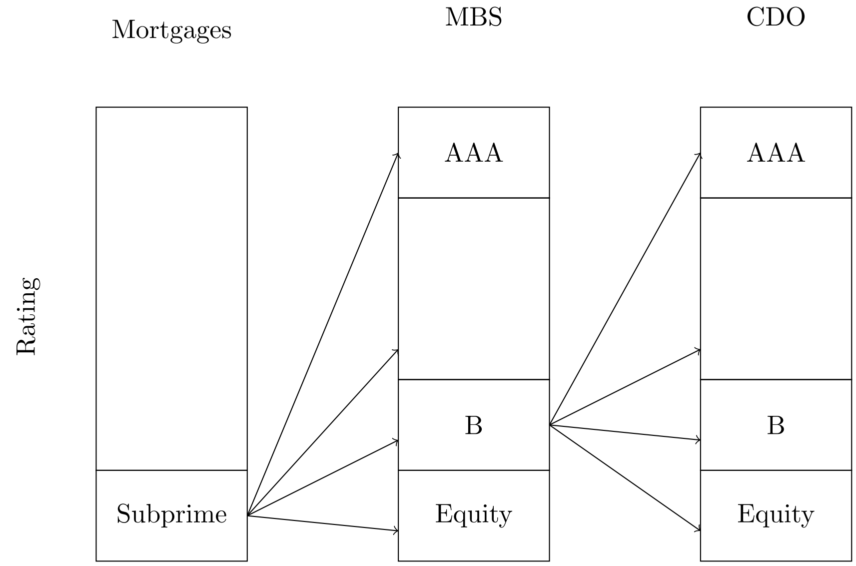


Figure 1: Securitizing subprime mortgages

This paper studies the effect of securitization on asset prices when there are heterogeneous beliefs in the market. This is achieved by comparing a benchmark scenario, where securitization is absent, with scenarios of increasing securitization. The underlying hypothesis is that the gradual introduction of asset backed securities (ABS), like MBS, CDO and squared CDO, inflated the underlying asset prices. Thanks to belief disagreement, securitization led asset prices to depart from fundamentals[1].

To formalize this idea, the paper develops a model in which, on each of many infinitely countable periods, risk-neutral agents trade assets of three kinds: a) an infinitely-supplied, risk-free, one-period bond b) a finitely-supplied, risky asset that pays off a dividend every period and c) securities backed by the latter. No naked positions are allowed, hence ownership of the long-term asset is necessary to issue the securities. These securities are of a limited liability nature: by transferring the long-term asset to a special purpose vehicle

[1]For a historical account of security innovation, see Matthews (1994).

(SPV), the issuer guarantees that only the investors who hold the securities will have any claims on the underlying asset. Importantly, it is assumed that there is an exogenous limit on the short-selling of the long-term asset and of any asset backed security.

The model of Harrison and Kreps (1978), in which there a no derivatives, serves as a benchmark for this paper. By assuming short-selling constraints and divergent beliefs, they show that the price of the long-term asset is given by the valuation of the most optimistic agent. Since the asset could be resold once the owner is no longer the most optimistic agent, the price surpassess any expectation regarding the present value of dividends. Eyster and Piccione (2013) use the same framework but provide more structure to agents beliefs. Their admissible beliefs can be represented as partitions of the state-space, formalizing the idea that people have a coarse understanding of their environment. This paper uses their class of admissible beliefs when studying the effect of securitization on average portfolio returns.

The way in which securitization is modeled is as follows. By securitizing the asset, the issuer of the securities commits to transfer the cash flow from the underlying asset to the investors who have bought the securities. The securities could simply be pass-trough securities in which the cash flow goes to investors in proportion to the securities bought. However, more sophisticated securities could be created by "slicing" or *tranching* the cash flow into tiers or *tranches*. The securities associated to the most senior tranche promise a fixed amount of cash unless the total cash flow from the underlying asset is below this amount. The next tranche entitles a fixed amount unless the total cash flow net of the senior tranche payoff is below this amount, and so on. This waterfall payment structure is also used by Malamud et al. (2010) and Grodecka (2013). For simplicity, the securities are assumed to have a maturity of one period.

Compared to the benchmark, securitization leads to an increase in the underlying asset price if and only if, in the benchmark scenario, no perceived distribution regarding the next-period price first-order stochastically dominates any other. If this is the case, tranching allows to create securities that are tailor-made to investors with different beliefs. Since ownership of the asset is necessary to issue these claims, the demand for the asset increases. As a consequence, the divergence between the asset price and any perceived present value of dividends increases. Furthermore, since different tranches could be bought by investors with different beliefs, each investor will think that the others are overpaying for theirs. Hence the whole asset would be traded at a price higher than anyone thinks it is worth.

Although securitization is modeled as a refinement of the existing tranching, it alternatively can be seen as the process of creating securities backed by asset backed securities. The paper shows that the two representations are equivalent. Therefore, the model implicitly captures what Geanakoplos (1996) denominates *pyramiding arrangements*: using the long-term asset as collateral, an agent may borrow from another agent, who in turn uses the issued debt as collateral to borrow from a third agent. In the original representation, this implies that mezzanine tranches are implicitly borrowing from senior tranches and lending to junior tranches.

In order to assess the effect of securitization on the rates of return received by agents, the paper uses the behavioral framework of Eyster and Piccione (2013). In their model, each agent is characterized by a partition of the state space. The finer the partition, the more sophisticated they are. Eyster and Piccione (2013) show that an agent might obtain an average return lower than the one obtained by a less sophisticated agent. This is because an heterogeneous and coarse understanding of the environment imply a form of winner's curse: buying the long-term asset means that all other agents are pessimistic about its payoff next-period. This paper shows that a consequence of this is that securitization could weakly decrease the returns agents receive.

This paper belongs to the branch of asset pricing literature under belief disagreement, of which Scheinkman and Xiong (2004) offer a survey. Milgrom and Stokey (1982) show that, if rational agents differ on information only but coincide on priors, there would be no trade. Because of this, numerous studies depart from the assumption of common priors. Miller (1977) and Harrison and Kreps (1978) were pioneers on this. A non-exhaustive list of other studies on asset pricing with heterogeneous beliefs includes Harris and Raviv (1993), Zapatero (1998), Chiarella and He (2001), Scheinkman and Xiong (2003), Cao and Ou-Yang (2009), Xiong and Yan (2010), Geanakoplos (2010), He and Xiong (2010), Cao (2011) and Hanson and Sunderam (2013). Empirically, Hong and Stein (2007) advocate for the use of heterogeneous beliefs to explain observed patterns in financial markets.

The paper also intersects with the literature on financial innovation and security design, of which Duffie and Rahi (1995) offer a survey. Recent examples include Brock et al. (2009), Che and Sethi (2010), Fostel and Geanakoplos (2012), Kubler and Schmedders (2012) and Simsek (2013a,b), which share with this paper the aim of studying the potentially destabilizing effects of financial innovation. In Brock et al. (2009), the expectations with a higher average profit are favorably selected by the market. The learning process might overshoot, though. Under incomplete markets, this implies that the more Arrow-Debreu securities are introduced, the more likely are the markets to become unstable. Kubler and Schmedders (2012) also study the effect of financial innovation under heterogeneous beliefs but in the context of an overlapping generations model. Their focus is on the effect of completing the markets on asset price volatility. Once markets become complete, wealth starts to shift across generations. As a consequence of the different propensities to consume between generations, asset price volatility increases. Simsek (2013b) shows that, under heterogeneous beliefs, the introduction of new assets could increase rather than decrease the average portfolio variance, as measured by the variance of agents net worth.

The works of Fostel and Geanakoplos (2012) and Simsek (2013a), in particular, are closely related to this paper. Both works build upon on a two-period general equilibrium model where agents diverge on their beliefs regarding the dividend of a physical asset. In Fostel and Geanakoplos (2012), there are only two states and a continuum of beliefs. In Simsek (2013a), there is a continuum of states but only two beliefs. Both papers work with the assumption that agents can not commit to pay what they promise. Therefore, creditors demand collateral in the form of either cash or the physical asset. This paper shares their prediction regarding the effect of tranching on asset prices, but in a context of partial equilibrium, infinite periods, any finite number of states and -for the most part- any finite number of beliefs. Furthermore, the aforementioned representation equivalence implies that the paper includes the case where promises themselves can be used as collateral. The paper also goes beyond the aim of these two works by studying the effect of securitization on the return each type of agent receives in the long run.

Another closely related paper to this is Garmaise (2001). His paper studies the problem of a firm designing a security to be auctioned among two investors. Investors do not have rational expectations: rather they use history to achieve statistical consistency, in a way similar to Eyster and Piccione (2013). Apart from this, the problem is essentially static. Furthermore, the firm is constrained to issue only one security. The optimal security maximizes differences in opinion, which is the same motivation found on this paper for an issuer to tranche as much as possible any asset they own.

The rest of this paper is organized as follows. Section 2 gives an introductory example which illustrates the setup and gives a hint of of the main results. Section 3 describes the general model, defines the equilibrium and proves its existence and uniqueness. Section 4 studies the effect of securitization on the long-term asset price and its relation to any perceived present value of dividends. Section 5 shows that multiple rounds of securitization are equivalent to a refinement of the initial tranching. Section 6 studies the effect of securitization on actual portfolio returns. Section 7 comments on relaxing the waterfall payment constraint. Finally section 8 presents some conclusions and suggestions for further research.

## 2. MOTIVATING EXAMPLE

Consider an environment where trade happens on each of infinite countable periods. Each period there is an infinitely-supplied bond used as a numeraire that entitles a payoff $R > 1$ next period. The bond could be traded for a finitely-supplied asset that yields a random dividend $d(\cdot)$ every period. The world could be in 3 possible states: $l$, $m$ and $h$. The realizations of the dividend are such that $d(l) < d(m) < d(h)$. Securities backed by the long-term asset may exist and traded as well.

There are two types of infinitely-lived agents, $\mathcal{A}$ and $\mathcal{B}$. Both are risk-neutral. In every state, agent $\mathcal{A}$ attaches a probability $1/3$ to transit to any other state. Agent $\mathcal{B}$ attaches a probability close to one to stay in the the current state. The disagreement may emerge as a consequence of agents using different forecasting tools or not using all information available. The motivation for heterogeneous beliefs is plenty, but it certainly requires at least some agents to depart from rational expectations: if information were used efficiently, the realizations of the stochastic process will feedback agents beliefs. Implicitly, the model assumes that the learning process has stalled.

### 2.1. *As in the benchmark scenario*

In the benchmark scenario, there are no asset backed securities. Because of limited short-selling, the market clears when the long-term asset highest expected return matches the return on the short-term bond. Let $q : \{l, m, h\} \to \mathbb{R}_+$ be the function that maps states of the world into asset prices. Notation is simplified by assuming that the asset price is *cum-dividend* and that the declared dividend is to be paid next period.

Consider the state $x \in \{l, m, h\}$. Both agents are willing to pay $d(x)/R$ plus the discounted asset price they expect for next period. For agent $\mathcal{A}$, the expected price is $\frac{1}{3}(q(l) + q(m) + q(h))$. For agent $\mathcal{B}$, it is $q(x)$. If, as it happens in equilibrium, $q(l) < q(m) < q(h)$, agent $\mathcal{A}$ buys the long-term asset asset in state $l$ whereas agent $\mathcal{B}$ does it in state $h$. Ex-ante, it is not clear who will buy the asset at state $m$. Still, the solution to the price function $q(\cdot)$ is given by

$$q(l) = \frac{d(l) + \frac{1}{3}(q(l) + q(m) + q(h))}{R} \tag{2.1}$$

$$q(m) = \frac{d(m) + \max\left\{q(m), \frac{1}{3}(q(l) + q(m) + q(h))\right\}}{R} \tag{2.2}$$

$$q(h) = \frac{d(h) + q(h)}{R} \tag{2.3}$$

For instance, if $R = 2$, $d(l) = 0$, $d(m) = 2$ and $d(h) = 3$, then $q(l) = 1$, $q(m) = 2$ and $q(h) = 3$.

To hint how securitization could increase the long-term asset price, consider the perceived distributions regarding the asset price for next period. Let $\underline{G}_i(t|x)$ be the probability that next-period asset price is less or equal than $t \in \mathbb{R}_+$ as perceived by agent $i \in \{\mathcal{A}, \mathcal{B}\}$, conditional on being on state $x \in \{l, m, h\}$. Figure 2 illustrates these conditional distributions for each of the three states. In states $l$ and $h$, the distributions can be ranked according to first-order stochastic dominance. The agents whose beliefs first-order stochastically dominate the others will buy the asset. At state $m$, there is no stochastic dominance between $\underline{G}_{\mathcal{A}}(\cdot|m)$ and $\underline{G}_{\mathcal{B}}(\cdot|m)$. Whoever has highest expectations regarding the asset price next period will buy the asset. At this state, however, agent $\mathcal{B}$ is more optimistic regarding the price not being below $q(m)$, whereas agent $\mathcal{A}$ is more optimistic regarding the price being above $q(m)$. This divergence is exploited by "slicing" the asset into two securities: one whose payoff is attached to realizations below $q(m)$ and the other to realizations above $q(m)$. This is illustrated as follows.

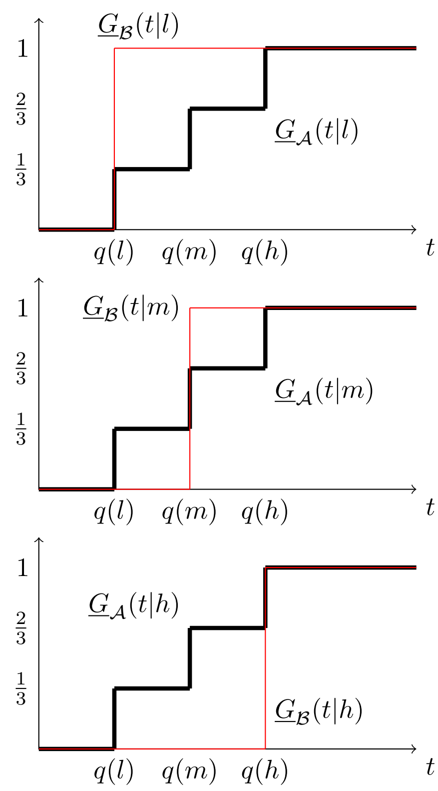


Figure 2: Perceived distributions on asset price

## 2.2. *As in a securitized scenario*

Assume now the any owner of the long-term asset can issue one-period maturity claims on the cash flow from the long-term asset. The claims add up the realization of the payoff from the long-term asset price. These claims are labeled asset backed securities. Naked positions are not allowed, so ownership of the asset is mandatory to issue these claims. The short selling of these securities is exogenously constrained.

Assume that the asset backed securities belong to one of two given tiers or "tranches". The *senior* tranche is relatively safe in the sense that guarantees certain payoff for next period, unless the realized state is bad ($l$). The *junior* tranche is risky in the sense that it only pays off if the realized state is good ($h$). Denote by $q' : \{l, m, h\} \to \mathbb{R}_+$ the asset price function in the securitized scenario. The senior tranche guarantees a payoff $q'(m)$ for next period unless the state becomes $l$, in which case it pays $q'(l)$. The payoff from the junior tranche is $q'(h) - q'(m)$ if the realized state is $h$, otherwise is 0. In short, if $\phi_s$ and $\phi_j$ denote the payoffs from senior and junior tranches as a function of the realization $q'(x)$, then

$$\phi_s(q'(x)) = q'(x) - (q'(x) - q'(m))^+ \tag{2.4}$$

$$\phi_j(q'(x)) = (q'(x) - q'(m))^+ \tag{2.5}$$

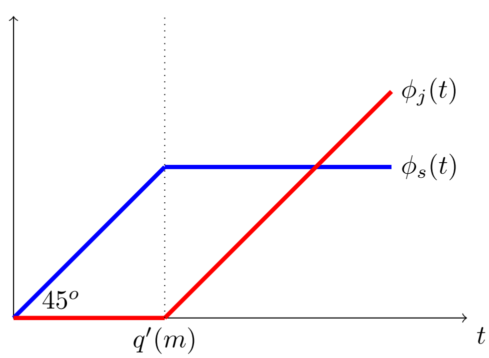


Figure 3: Payoff function from senior and junior tranches

Figure 3 plots the payoff from each tranche as a function of the cum-dividend price of the underlying asset. Note that i) owning the junior tranche is the same as having a one-period purchased call option on the underlying asset with strike price $q'(m)$ ii) owning the senior tranche is equivalent to owning the underlying asset entangled with a one-period written put option with strike price $q'(m)$. In this particular example, the size of each tranche is given by $q'(m)$. Such a threshold is known as an *attachment point*.

On an alternative interpretation, the buyer of the junior tranche becomes the owner of the asset, but has a debt liability to the buyer of the senior tranche and is using the underlying asset as collateral. In this example, the junior buyer promises to pay $q'(m)$ to the senior buyer next period. However, if the value of the collateral is $q'(l)$, the junior buyer defaults and the senior buyer seizes the collateral.

Since the short selling of the securities is limited, the price of each of them is given by the highest expected payoff from it, discounted by the interest rate. To start with, assume $q'(l) < q'(m) < q'(h)$. At state $m$, agent $\mathcal{B}$ expects a payoff $q'(m)$ from the senior tranche, while agent $\mathcal{A}$ expects $\frac{2}{3}q'(m)+\frac{1}{3}q'(l)$. Hence the senior tranche will be bought by agent $\mathcal{B}$. Meanwhile, agent $\mathcal{B}$ expects a zero payoff from the junior tranche, while agent $\mathcal{A}$ expects $q'(h) - q'(m)$ with one-third probability. Hence agent $\mathcal{A}$ will buy the junior tranche. If $p(s,x)$ and $p(j,x)$ denote the prices of senior and junior tranches at state $x \in \{l,m,h\}$, then

$$
\begin{aligned}
p(s,m) &= \frac{q'(m)}{R} \\
p(j,m) &= \frac{\frac{1}{3}\left(q'(h) - q'(m)\right)}{R}
\end{aligned}
$$

Since different agents might have claims on the same asset, securitization could break the equivalence between any single expectation regarding the value of the underlying asset and the total value of the securities. To show this, note that at state $m$, the expectation on the asset price for next period is $q'(m)$ for agent $\mathcal{B}$ and $\frac{1}{3}(q'(l)+q'(m)+q'(h))$ for agent $\mathcal{A}$. Then

$$
\begin{aligned}
p(s,m) + p(j,m) &> \frac{q'(m)}{R} \\
p(s,m) + p(j,m) &> \frac{\frac{1}{3}\left(q'(l) + q'(m) + q'(h)\right)}{R}
\end{aligned}
$$

which means that the payoff from selling the securities surpassess any expectation regarding the discounted price of the underlying asset[2].

The equilibrium price for the long-term asset is given by a non-arbitrage condition. Any agent could buy the long-term asset and sell the asset backed securities derived from it. In equilibrium, the long-term asset cum-dividend price has to equal the discounted declared dividend plus the market value of the asset backed securities:

$$q'(x) = \frac{d(x)}{R} + p(s,x) + p(j,x) \tag{2.6}$$

If the left-hand side of (2.6) were greater than the right hand side, there would be no demand for the long-term asset. On the other hand, if the left-hand side were lesser than the right-hand side, the demand would be infinite.

For states $l$ and $h$, the equilibrium conditions can be derived by identifying who buys the tranches at these states. At state $l$, the expectations of $\mathcal{A}$ on the payoff from any tranche are greater than those of $\mathcal{B}$. The opposite is true at state $h$. Hence both tranches are bought by $\mathcal{A}$ at state $l$ and both tranches are bought by $\mathcal{B}$ at state $h$. Adding up the tranche prices yields the discounted long-term asset price expected by whoever buys both tranches. Hence the price equations for $q'(l)$ and $q'(h)$ resemble equations (2.1) and (2.3). The same is not true for $q'(m)$, however, since at state $m$ each tranche is bought by different agents. In summary, the equilibrium price function $q'$ is given by

$$q'(l) = \frac{d(l) + \frac{1}{3}\left(q'(l) + q'(m) + q'(h)\right)}{R} \tag{2.7}$$

$$q'(m) = \frac{d(m) + q'(m) + \frac{1}{3}(q'(h) - q'(m))}{R} \tag{2.8}$$

$$q'(h) = \frac{d(h) + q'(h)}{R} \tag{2.9}$$

By comparing equation (2.8) with equation (2.2), it can be seen that securitization has increased the asset price at state $m$. This in turns increases the price at state $l$, since $\mathcal{A}$'s expectation at state $l$ is higher. Going back to the numerical example where $R = 2$, $d(l) = 0$, $d(m) = 2$ and $d(h) = 3$, it can be show that the asset price increases by $\frac{1}{20}$ at state $l$ and by $\frac{1}{4}$ at state $m$. Thanks to the reselling opportunity, securitization increases the asset price on states in which securitization appears to be innocuous.

[2]See Fostel and Geanakoplos (2012) for a similar result.

## 3. THE MODEL

In this section, the motivating example is extended to any finite number of states, any finite number of types of agents and any securitization with a waterfall payment structure. All agents perceive the stochastic process on dividends and interest rates as an irreducible Markov chain, but they may disagree on the transition probabilities. To obtain certain results, it sometimes will be assumed that agents beliefs satisfy the following condition: expectations one-period forward, averaged across states, match the true long-frequency of the random variable to be forecast. Note, however, that this does not imply that expectations have to be homogeneous.

### 3.1. *Dividends and interest rates*

There are infinite countable periods. On each of them, risk-neutral agents trade a) an infinitely-supplied, one-period-maturity bond b) a finitely-supplied asset that generates a dividend every period and c) some securities backed by the latter. The short-selling of the asset backed securities and of the long-term asset itself is exogenously constrained.

The background structure of the model is given by an exogenous stochastic process on the bond interest rate and on the long-term asset dividend. The state-space, denoted by $X$, is assumed to be finite. Buying a bond when the state is $x \in X$ entitles a payoff $R(x) > 1$ next period. Buying the long-term asset when the state is $x \in X$ entitles a dividend $d(x) \geq 0$ next period and the right to keep, sell or securitize the asset.

The state of the world is publicly observed and evolves according to a Markov process. The probability of transition from state $x \in X$ to state $y \in X$ is given by $P(x, y)$, which is assumed to be strictly positive for every pair of states. This assumption is sufficient for the existence of a function $\mu : X \to (0, 1)$ such that

$$\mu(y) = \sum_{x \in X} P(x, y)\mu(x)$$

for every $y \in X$. In the long-run, the world spends a fraction $\mu(x)$ of the time at state $x \in X$, regardless of the initial state.

## 3.2. *Beliefs*

Each agent has a theory which represents their beliefs regarding the behavior of dividends, interest rates and prices. The set of theories in the market is represented by $\mathbf{C}$ and is assumed to be finite. Denote by $Q_{\mathcal{F}}(x,y)$ the transition probability from state $x \in X$ to state $y \in X$ perceived by theory $\mathcal{F} \in \mathbf{C}$. An agent is said to have rational expectations if $Q_{\mathcal{F}} = P$.

Consider a random variable $g : X \to \mathbb{R}$ which could be a price, dividend, interest rate or any function of them., The expectation of agent $\mathcal{F}$ on $g$ one period forward, conditional on $x \in X$, is denoted by

$$E_{\mathcal{F}}\left(g\right)\left(x\right) := \sum_{y \in X} g(y) Q_{\mathcal{F}}\left(x,y\right)$$

Most of the results in this paper are agnostic regarding the source of heterogeneity of beliefs. Henceforth, the true stochastic process $P$ has no role within these results. However, sometimes the performance of each theory is compared with the truth, which should be interpreted as the long-term outcome. In this case, beliefs satisfy the constraints of Eyster and Piccione (2013). In this framework, each theory $\mathcal{F} \in \mathbf{C}$ is equivalent to a partition of $X$. An element of $\mathcal{F}$ is known as a block of theory $\mathcal{F}$. If $x$ and $z$ are in the same block, then the transition probabilities perceived by theory $\mathcal{F}$ at state $x$ are the same as when the state is $z$. In particular, for $x, y \in X$,

$$Q_{\mathcal{F}}\left(x,y\right) := \frac{\sum_{z \in \mathcal{F}(x)} P(z,y) \mu\left(z\right)}{\sum_{z \in \mathcal{F}(x)} \mu\left(z\right)} \tag{3.1}$$

where $\mathcal{F}(x)$ is the block in $\mathcal{F}$ that contains $x$. Note that if every block in $\mathcal{F}$ is a singleton, then an agent with theory $\mathcal{F}$ has rational expectations. On the other hand, if $\mathcal{F} = \{X\}$, then $Q_{\mathcal{F}}\left(x,y\right) = \mu(y)$ for all $x \in X$. In this framework, the long-run average of any random variable $g : X \to \mathbb{R}$ can be obtained by taking the expectation for $g$ next period by any $\mathcal{F} \in \mathbf{C}$ and averaging across states:

$$\sum_{x \in X} \left(E_{\mathcal{F}}\left(g\right)\left(x\right)\right) \mu(x) = \sum_{y \in X} g(y) \mu(y)$$

In this case, all agents achieve some degree of statistical consistency despite not necessarily having rational expectations.

Consider the motivating example of section 2. The set of beliefs was given by $\mathbf{C} = \{\mathcal{A}, \mathcal{B}\}$. The transition probabilities were given by $Q_{\mathcal{A}}(x, y) = \frac{1}{3}$ and $Q_{\mathcal{B}}(x, x) = 1 - \epsilon$ for $x, y \in X := \{l, m, h\}$. The case studied corresponded to the limiting case where $\epsilon \to 0$. The example was agnostic regarding the true Markov process $P : X^2 \to (0, 1)$. However, if $P(x, x) = 1 - \epsilon$ and $P(x, y) = \epsilon/2$ for $x, y \in X$ and $x \neq y$, the beliefs for $\mathcal{A}$ and $\mathcal{B}$ can be represented as the partitions $\{\{l, m, h\}\}$ and $\{\{l\}, \{m\}, \{h\}\}$, respectively. Agent $\mathcal{A}$'s expectations, despite not being rational, achieve the aforementioned statistical consistency.

### 3.3. *Securitization*

The ownership of the long-term asset entitles the possibility of obtaining liquidity (bonds) not only by reselling the asset but also through its *securitization*. It is assumed that the derived securities have a maturity of one-period. Each security is associated with a particular interval or *tranche*. The security is a contract in which the owner of the long-term asset (the issuer) commits to transfer the holder of the security (the investor) a (non-negative) amount of bonds next period. This amount depends on the realization of the asset price relative to the tranche when payment is due. The securities are backed by the long term asset in the sense that the net transfers from the the issuer to all investors are always equal to the payoff of the underlying asset.

Formally, let $\mathcal{T}$ be a collection of intervals that partition the set $[0, \infty)$, which is the set of all conceivable realizations for the asset price. The partition $\mathcal{T}$ is known as a *tranching* and an interval $\tau \in \mathcal{T}$ is known as a *tranche*. Define $\underline{\tau} := \inf(\tau)$ and $\overline{\tau} := \sup(\tau)$. The payoff function $\phi_\tau : \mathbb{R}_+ \to \mathbb{R}_+$ from a security associated with tranche $\tau$ is given by

$$\phi_\tau(t) := (t - \underline{\tau})^+ - (t - \overline{\tau})^+ \tag{3.2}$$

where $t \in \mathbb{R}_+$ is a realization for the underlying asset price .

The point $\underline{\tau}$ is known as the *attachment point* of tranche $\tau$. This tranche only starts to pay off when the realization for the asset price is above $\underline{\tau}$. Above $\overline{\tau}$, the payoff from tranche $\tau$ stalls at $\overline{\tau} - \underline{\tau}$ and the payoff from the next junior tranche (if any) deattaches from 0. Figure 4 illustrates the payoff function for a generic tranche (also known as a straddle).

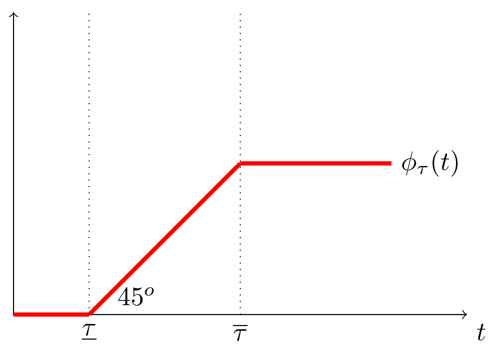


Figure 4: Payoff function of a generic tranche $\tau$

The securities are backed up by the long-term asset in the sense that they are *budget balanced*: if $t$ is the realization of the asset price at the expiration date, the tranches payoffs add up to $t$:

$$\sum_{\tau \in \mathcal{T}} \phi_\tau(t) = t$$

In addition, there is *limited liability* for the issuer: $\phi_\tau(t) \geq 0$ for all $\tau \in \mathcal{T}$ and $t \in \mathbb{R}_+$.

In the example of subsection 2.1, there was no securitization so the tranching was given by $\mathcal{T} = \{[0, \infty)\}$. In subsection 2.2, there were only two tranches and the attachment point was calibrated so that the tranching was $\mathcal{S} = \{[0, q'(m)), [q'(m), \infty)\}$. The senior tranche was $s = [0, q'(m))$ while the junior tranche was $j = [q'(m), \infty)$. Hence the payoff functions in equations (2.4) and (2.5) could be equivalently defined by using the expression in (3.2).

Beneath the tranche payoff function there is the assumption that, before trade is carried out, agents know the dividend to be paid next period. Securatizing the dividend in addition the principal $q(\cdot)$ would be innocuous for asset pricing if all tranches are adjusted accordingly. All results can be extended if the tranching is state-contingent.

### 3.4. *Equilibrium*

In equilibrium, the demand for the long-term asset and the demand for each security have to match their supply. The exogenous limit on short-selling guaranties that this supply is finite. For $x \in X$, let $p(\tau, x)$ be the price of the security associated with tranche $\tau \in \mathcal{T}$ and $q(x)$ be the price of the long-term asset.

DEFINITION 3.1 *For a given tranching $\mathcal{T}$, a waterfall equilibrium is a pair of functions $q : X \to \mathbb{R}_+$ and $p : \mathcal{T} \times X \to \mathbb{R}_+$ such that*

$$q(x) = \frac{d(x)}{R(x)} + \sum_{\tau \in \mathcal{T}} p(\tau, x) \tag{3.3}$$

*and*

$$p(\tau, x) = \frac{\max_{\mathcal{F} \in \mathbf{C}} E_{\mathcal{F}} \left( (q - \underline{\tau})^+ - (q - \overline{\tau})^+ \right)(x)}{R(x)} \tag{3.4}$$

*for $x \in X$ and $\tau \in \mathcal{T}$.*

Equation (3.3) states that, in equilibrium, zero profits are made by buying the long-term asset and selling the securities derived from it. Equation (3.4) states that the price of each security is given by the highest discounted expectation regarding the payoff from the corresponding tranche.

### 3.5. *Existence and uniqueness*

The existence and uniqueness of a waterfall equilibrium is proved by constructing the following object. For any tranching $\mathcal{T}$, define a mapping $\Psi_{\mathcal{T}}$ from the set of price functions $\mathbb{R}_+^X$ to itself. For $q \in \mathbb{R}_+^X$ and $x \in X$, the mapping is given by

$$\Psi_{\mathcal{T}}(q)(x) = \frac{d(x) + \sum_{\tau \in \mathcal{T}} \max_{\mathcal{F} \in \mathbf{C}} E_{\mathcal{F}} \left( (q - \underline{\tau})^+ - (q - \overline{\tau})^+ \right)(x)}{R(x)} \tag{3.5}$$

By replacing equation (3.4) into equation (3.3), it follows that a long-term asset price function $q$ must be a fixed point of $\Psi_{\mathcal{T}}$ in order to be part of an equilibrium.

Proposition 3.1 *For every tranching $\mathcal{T}$, there is a unique waterfall equilibrium.*

Proof: The mapping $\Psi_{\mathcal{T}}$ is a contraction mapping and therefore has a unique fixed point since it satisfies Blackwell's sufficient conditions:

1. (monotonicity) For all $q, q' \in \mathbb{R}_+^X$ such that $q(x) \leq q'(x)$ for all $x \in X$,

$$\Psi_{\mathcal{T}}(q)(x) \leq \Psi_{\mathcal{T}}(q')(x)$$

for all $x \in X$.

2. (discounting) For any $q \in \mathbb{R}_+^X$ and $c \geq 0$,

$$\Psi_{\mathcal{T}}(q+c)(x) \leq \Psi_{\mathcal{T}}(q)(x) + \left(\min_{y \in X} R(y)\right)^{-1} c$$

for all $x \in X$.

*Q.E.D.*

## 4. EFFECTS OF SECURITIZATION

This section studies the effect of securitization on the equilibrium price for the long-term asset. Further securitization is understood as a refinement of a tranching. This means that an issuer is generating more tranches when securitizing an asset. It is shown in section 5 that this is equivalent to issuing new securities backed by asset backed securities.

### 4.1. *Effect on asset prices*

Denote by $q_{\mathcal{T}}$ the waterfall equilibrium price for the long-term asset for tranching $\mathcal{T}$. Further securitization cannot decrease the price of the underlying asset:

Lemma 4.1 *If $\mathcal{S}$ is a refinement of $\mathcal{T}$, then*

$$q_{\mathcal{T}}(x) \leq q_{\mathcal{S}}(x)$$

*for all $x \in X$*

PROOF: See appendix.

Intuitively, there is no harm to the issuer in slicing the cash flow from the underlying asset into more tranches: if lucky, investors who were outbided when issuing big tranches would push harder when bidding for smaller tranches. At worst, investors would be indifferent.

An equilibrium of particular interest is the benchmark without securitization. For this case, the equilibrium price function is given by $\underline{q} := q_{\{[0,\infty)\}}$. Since any tranching $\mathcal{T}$ is a refinement of $\{[0,\infty)\}$, it follows from Lemma 4.1 that

$$\underline{q}(x) \leq q_{\mathcal{T}}(x) \tag{4.1}$$

for $x \in X$. The function $\underline{q}$ is a lower bound on asset prices in the sense that any tranching $\mathcal{T}$ cannot decrease the asset price below $\underline{q}(x)$ at any $x \in X$. Adding up the willingness to pay for all tranches can not be below the willingness to pay for the whole asset.

Still, a satisfactory explanation of securitization will have to shown that it *strictly* increases the asset price in at least one state. If this is the case, it seems natural to assume that any issuer will securitize the asset until securitization stops increasing the asset value. A tranching is labeled *issuer-optimal* if no other tranching can increase the asset price at any state:

DEFINITION 4.1 *A tranching $\mathcal{T}$ is issuer-optimal if*

$$q_{\mathcal{S}}(x) \leq q_{\mathcal{T}}(x)$$

*for any tranching $\mathcal{S}$ and all $x \in X$.*

In order to characterize the necessary and sufficient conditions for a tranching to be issuer-optimal, the following object is now defined. Let $I_A$ be the indicator function for event $A$. For $t \in \mathbb{R}_+$, define $G^{\mathcal{T}}_{\mathcal{F}}(t|x)$ as the probability at state $x$ that the asset price next period is no higher than $t$, as perceived by theory $\mathcal{F}$ when the tranching is $\mathcal{T}$:

$$G^{\mathcal{T}}_{\mathcal{F}}(t|x) := E_{\mathcal{F}}\left(I_{\{q_{\mathcal{T}} \leq t\}}\right)(x)$$

Theorem 4.1 *A tranching $\mathcal{T}$ is issuer-optimal if and only if, for each $\tau \in \mathcal{T}$ and $x \in X$, there is a theory $\mathcal{G} \in \mathbf{C}$ such that*

$$G_{\mathcal{G}}^{\mathcal{T}}(t|x) \leq G_{\mathcal{F}}^{\mathcal{T}}(t|x)$$

*for all $t \in \tau$ and all $\mathcal{F} \in \mathbf{C}$.*

Proof: See appendix.

Theorem 4.1 states that a tranching is issuer-optimal if and only if for every state and every tranche, there is someone who is more optimistic than anyone else regarding the asset price not being below any realization along the tranche. If this is not the case, there is a tranching that will increase the asset price in every state. This is because at some state there will be someone willing to outbid whoever is buying the current tranche whenever a smaller tranche is issued.

A natural question is whether the benchmark with no tranching is issuer-optimal. Denote by $\underline{G}_{\mathcal{F}}(\cdot|x)$ the cumulative distribution function on the asset price for theory $\mathcal{F}$ when there is no tranching, i.e,

$$\underline{G}_{\mathcal{F}}(t|x) := E_{\mathcal{F}}\left(I_{\left\{\underline{q} \leq t\right\}}\right)(x)$$

for $t \in \mathbb{R}_{+}$.

Corollary 4.1 *The tranching $\{[0, \infty)\}$ is issuer-optimal if and only if, for each $x \in X$, there is a theory $\mathcal{G} \in \mathbf{C}$ such that $\underline{G}_{\mathcal{G}}(\cdot|x)$ first-order stocastically dominates $\underline{G}_{\mathcal{F}}(\cdot|x)$ for all $\mathcal{F} \in \mathbf{C}$.*

Proof: Follows from theorem 4.1 *Q.E.D.*

If there is only one theory in the market, the first-order stochastic dominance condition trivially holds. As expected, the benchmark without tranching is issuer-optimal if there is no belief disagreement.

As an example of a tranching that is not issuer-optimal, consider $\{[0,\infty)\}$ in subsection 2.1: there is no first-order stochastic dominance between beliefs at state $m$, as illustrated by Figure 2. On the other hand, the tranching $\{[0, q'(m)), [q'(m), \infty)\}$ in subsection 2.2 is issuer-optimal: for each tranche and each state there is always an agent whose conditional cumulative distribution on the asset price caps the other one along the realizations in the tranche.

Finally, it is worth remarking that if securitization increases the asset price, the asset would be traded at the price higher than anyone thinks it is worth:

REMARK 1 *If $\{[0,\infty)\}$ is not issuer-optimal, then there is a tranching $\mathcal{T}$ in which the waterfall equilibrium $q_{\mathcal{T}}$ is such that*

$$q_{\mathcal{T}}(x) > \frac{d(x) + \max_{\mathcal{F}\in\mathbf{C}} E_{\mathcal{F}}\left(q_{\mathcal{T}}\right)(x)}{R(x)}$$

*in at least one state $x \in X$.*

PROOF: By Definition 4.1, there is a tranching $\mathcal{T}$ such that

$$\underline{q}(x) \leq q_{\mathcal{T}}(x)$$

with strict inequality for at least one $x \in X$.

Suppose that

$$q_{\mathcal{T}}(x) \leq \frac{d(x) + \max_{\mathcal{F}\in\mathbf{C}} E_{\mathcal{F}}\left(q_{\mathcal{T}}\right)(x)}{R(x)}$$

for every $x \in X$. Since

$$\frac{d(x) + \max_{\mathcal{F}\in\mathbf{C}} E_{\mathcal{F}}\left(q_{\mathcal{T}}\right)(x)}{R(x)} \leq \Psi_{\mathcal{T}}\left(q_{\mathcal{T}}\right)$$

then

$$q_{\mathcal{T}}(x) = \frac{d(x) + \max_{\mathcal{F}\in\mathbf{C}} E_{\mathcal{F}}\left(q_{\mathcal{T}}\right)(x)}{R(x)}$$

for every $x \in X$. But if that were the case, then both $q_{\mathcal{T}}$ and $\underline{q}$ would be fixed points of $\Psi_{\{[0,\infty)\}}$, contradicting uniqueness. *Q.E.D.*

Intuitively, in an equilibrium different from the no-securitization benchmark, there has to be at least one state in which the tranching allows for an investor whom otherwise would not have bought the whole asset to buy a piece of it. Since investors with different theories will be buying different tranches of the same asset, each of them will think that the others are overpaying for theirs.

### 4.2. *Relation to Fundamentals*

Consider the relationship between the asset price and the perceived fundamental values, namely the present value of dividends perceived by each theory in the market. Let $v_{\mathcal{F}}(x)$ be the present value of dividends perceived by $\mathcal{F} \in \mathbf{C}$ at sate $x \in X$. This value function is given by the recursive solution to

$$v_{\mathcal{F}}(x) = \frac{d(x) + E_{\mathcal{F}}\left(v_{\mathcal{F}}\right)(x)}{R(x)} \tag{4.2}$$

for $x \in X$.

Proposition 4.1 *For any tranching $\mathcal{T}$ and collection of theories $\mathbf{C}$,*

$$v_{\mathcal{F}}(x) \leq q_{\mathcal{T}}(x)$$

*for all $x \in X$ and $\mathcal{F} \in \mathbf{C}$. Furthermore, if $\{[0,\infty)\}$ is not issuer-optimal, then there is a tranching $\mathcal{T}$ such that*

$$v_{\mathcal{F}}(x) < q_{\mathcal{T}}(x)$$

*for all $x \in X$ and $\mathcal{F} \in \mathbf{C}$.*

Proof: The definition of waterfall equilibrium for the benchmark case

$$\underline{q}(x) = \frac{d(x) + \max_{\mathcal{F} \in \mathbf{C}} E_{\mathcal{F}}\left(\underline{q}\right)(x)}{R(x)}$$

and equation (4.2) imply

$$v_{\mathcal{F}}(x) \leq \underline{q}(x) \tag{4.3}$$

for all $x \in X$ and $\mathcal{F} \in \mathbf{C}$. The first claim follows from inequality (4.3) and inequality (4.1). The second claim follows from inequality (4.3) and Proposition 9.2 in the appendix. *Q.E.D.*

Proposition 4.1 states that if the benchmark equilibrium without securitization is not issuer-optimal, then further securitization will eventually increase the gap between the asset price and any perceived present value of dividends. Because of reselling opportunities, securitization increases this gap at every state.

## 5. ITERATED SECURITIZATION

In section 4, increased securitization has been modeled as a refinement of the existing tranching, which is equivalent to create more tranches for the cash flow coming from the long-term asset. Securitization, however, is also seen as issuing securities backed by asset backed securities, as when CDOs are created based on existing MBS or squared CDOs are created based on existing CDOs. This section shows the equivalence between these two interpretations. Remarkably, any tranching is equivalent to issuing debt using debt securities as collateral and iterating this process a finite number of times.

### 5.1. *Example*

Consider the case where $X = \{a, b, c, d\}$ and the probability of a state going back to itself is close to one and uniform otherwise. Hence the system spends a quarter of the time on each state. The interest rate is assumed to be constant. The collection of theories in the market is given by $\mathbf{C} = \{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$, where

$$\begin{aligned} \mathcal{A} &= \{\{a, d\}, \{b, c\}\} \\ \mathcal{B} &= \{\{a, b, d\}, \{c\}\} \\ \mathcal{C} &= \{\{a, b, c, d\}\} \end{aligned}$$

All theories in the market achieve statistical consistency: the perceived transition probability for agent $\mathcal{A}$ from states $a$ or $d$ to states $a$ or $d$ tends to one half, the perceived transition probability for agent $\mathcal{B}$ from states $a$, $b$ or $d$ to states $a$, $b$ or $d$ tends to one third

TABLE I

Security payoffs for the issuer-optimal tranching

| States | $\phi_e$ | $\phi_j$ | $\phi_m$ | $\phi_s$ |
|---|---|---|---|---|
| $a$ | $q(a)-q(b)$ | $q(b)-q(c)$ | $q(c)-q(d)$ | $q(d)$ |
| $b$ | 0 | $q(b)-q(c)$ | $q(c)-q(d)$ | $q(d)$ |
| $c$ | 0 | 0 | $q(c)-q(d)$ | $q(d)$ |
| $d$ | 0 | 0 | 0 | $q(d)$ |

and the perceived transition probability for agent $\mathcal{C}$ from any state to any other tends to one fourth.

A tranching that is issuer-optimal is one where there are four tranches and the attachment points coincide with the realizations of the asset price. Let $q$ be the waterfall equilibrium price function for the issuer-optimal tranching. Assume that the dividend function is such that $q(a) > q(b) > q(c) > q(d)$. In order of seniority, the tranches are labeled $s$ (senior), $m$ (mezzanine), $j$ (junior) and $e$ (equity). Table I presents the payoff function for each tranche.

At state $d$, theory $\mathcal{A}$ buys the tranche $e$, theory $\mathcal{B}$ buys the tranche $j$ and theory $\mathcal{C}$ buys the tranche $m$. All agents have the same willingness to pay regarding tranche $s$. At state $d$, three types of agents have differentiated claims on the underlying asset.

Instead of a unique round of securitization with four tranches, consider three rounds of securitization. In the first round, the underlying asset is securitized into debt and equity tranches. In the second round, the debt tranche of the first round is securitized into other debt and equity tranches. In the third round, the debt tranche of the second round is securitized into more debt and equity tranches (Figure 5). Let $\phi_e^i$ and $\phi_f^i$ be the payoffs from the equity and the debt tranches in the $i$-th round of securitization, where $i \in \{1, 2, 3\}$. Table II shows the assumed payoff for each tranche. By construction, the payoff from the debt and the equity tranches add up the payoff from the debt tranche from the previous round.

Consider the equilibrium at state $d$. The equity tranche of the 1st round will be bought by theory $\mathcal{A}$, the equity tranche of the 2nd round will be bought by theory $\mathcal{B}$, and the eq-

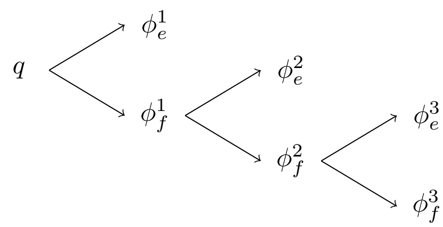


Figure 5: Iterated securitization

TABLE II

Security payoffs for the iterated case

| States | $\phi_e^1$ | $\phi_f^1$ | $\phi_e^2$ | $\phi_f^2$ | $\phi_e^3$ | $\phi_f^3$ |
|---|---|---|---|---|---|---|
| $a$ | $q(a)-q(b)$ | $q(b)$ | $q(b)-q(c)$ | $q(c)$ | $q(c)-q(d)$ | $q(d)$ |
| $b$ | 0 | $q(b)$ | $q(b)-q(c)$ | $q(c)$ | $q(c)-q(d)$ | $q(d)$ |
| $c$ | 0 | $q(c)$ | 0 | $q(c)$ | $q(c)-q(d)$ | $q(d)$ |
| $d$ | 0 | $q(d)$ | 0 | $q(d)$ | 0 | $q(d)$ |

uity tranche of the 3rd round will be bought by theory $\mathcal{C}$. The value of the debt tranche on the 3rd round will be $q(d)$, discounted by the interest rate. Given this and the willingness to pay by agent $\mathcal{C}$ on the equity tranche of the 3rd round, the value of the debt tranche in the 2nd round will be $q(d)$ plus three quarters of $q(c)-q(d)$, discounted by the interest rate. Given this and the willingness to pay by agent $\mathcal{B}$ on the equity tranche of the 2nd round, the value of the debt tranche in the 1st round will be $q(d)$ plus three quarters of $q(c)-q(d)$ plus two thirds of $q(b)-q(c)$, discounted by the interest rate. Given this and the willingness to pay by agent $\mathcal{A}$ on the equity tranche of the 1st round, the value of the asset at state $d$ will be the dividend at $d$ plus $q(d)$ plus three quarters of $q(c)-q(d)$ plus two thirds of $q(b)-q(c)$ plus half of $q(a)-q(c)$, discounted by the interest rate. This is the same asset price equation for the aforementioned case of one round of securitization with four tranches.

A way of reading the equilibrium transactions at state $d$ is that agent $\mathcal{A}$ is borrowing from agent $\mathcal{B}$ using the underlying asset as collateral. In turn, agent $\mathcal{B}$ is borrowing from agent $\mathcal{C}$ using the debt issued by $\mathcal{A}$ as collateral. The equilibrium features a *pyramiding arrangement*, as defined by Geanakoplos (1996).

### 5.2. *General Case*

In order to define what would be labeled an *iterated securitization equilibrium*, consider the following structure. Let there be a tree where $\mathbb{N}$ is its finite set of nodes, $r \in \mathbb{N}$ is its root node and $\mathbb{Z} \subset \mathbb{N}$ is its set of terminal nodes. Let $a : \mathbb{N}/\{r\} \to \mathbb{N}/\mathbb{Z}$ be a function where $a(n)$ denotes the predecessor of $n$. The root node $r$ represents the long-term asset. Each node $n \in \mathbb{N}/\{r\}$ represents a security backed by asset $a(n)$. Nodes in $\mathbb{Z}$ represent securities that are not further securitized.

For $m \in \mathbb{N}/\mathbb{Z}$, denote by $S(m)$ the set of $m$'s successors:

$$S(m) = \{n \in \mathbb{N}/\{r\} : a(n) = m\}$$

Let $(\tau_n)_{n\in\mathbb{N}}$ be a collection of intervals such that

1. $\tau_r = [0, \infty)$ and
2. for $m \in \mathbb{N}/\mathbb{Z}$, the collection $(\tau_n)_{n\in S(m)}$ partitions the interval

$$[0, \sup(\tau_m) - \inf(\tau_m)]$$

Each security in $\mathbb{N}/\{r\}$ is a straddle with a maturity of one period. Its payoff is a function of the cash flow from its predecessor. Specifically, if $t$ is the cash flow from asset $a(n)$, then the cash flow for security $n \in \mathbb{N}/\{r\}$ would be given by

$$\psi_n(t) = (t - \inf(\tau_n))^+ - (t - \sup(\tau_n))^+ \tag{5.1}$$

Hence, for $m \in \mathbb{N}/\mathbb{Z}$,

$$\psi_m(t) = \sum_{n\in S(m)} \psi_n(t)$$

confirming that the cash flow from securities in $S(m)$ add up the cash flow from asset $m$.

Let $\gamma_n : X \to \mathbb{R}_+$ be the cash flow for security $n \in \mathbb{N}$ as a function of the state of the world. If $q : X \to \mathbb{R}_+$ is the price function for the long-term asset, then $\gamma_r = q$ and

$$\gamma_n(x) = \psi_n(\gamma_{a(n)}(x)) \tag{5.2}$$

for $n \in \mathbb{N}/\{r\}$ and $x \in X$.

A tuple $\Sigma = (\mathbb{N}, r, \mathbb{Z}, a, (\tau_n)_{n\in\mathbb{N}})$ is labeled an *iterated securitization structure*.

Definition 5.1 *For a given iterated securitization structure $\Sigma$, an iterated securitization equilibrium is a function $p_\Sigma : \mathbb{N} \times X \to \mathbb{R}_+$ such that, for $x \in X$,*

*1.*

$$p_\Sigma(r, x) = \frac{d(x)}{R(x)} + \sum_{n \in S(r)} p_\Sigma(n, x) \tag{5.3}$$

*2. For $m \in \mathbb{N}/(\mathbb{Z} \cup \{r\})$,*

$$p_\Sigma(m, x) = \sum_{n \in S(m)} p_\Sigma(n, x) \tag{5.4}$$

*3. For $z \in \mathbb{Z}$,*

$$p_\Sigma(z, x) = \frac{\max_{\mathcal{F} \in \mathbf{C}} E_{\mathcal{F}}\left(\gamma_z\right)(x)}{R(x)} \tag{5.5}$$

*where $\gamma_r = p_\Sigma(r, \cdot)$ and, for $n \in \mathbb{N}/\{r\}$, $\gamma_n$ is given by equation (5.2).*

The function $p_\Sigma(r, \cdot)$ is the price function for the long-term asset. Equation (5.3) states that it is given by the discounted declared dividend plus the added prices of the securities derived from it. Equation (5.4) states that for the securities that are not terminal nodes, their price is given by the added prices of the securities derived from them. Finally Equation (5.5) gives the equilibrium for the terminal nodes, i,e., the securities that are not further securitized. Their price is given by the highest expectation regarding the cash flow coming from them. Ultimately, the cash flow from the securities at the terminal nodes will depend on the price of the long-term asset next period.

Theorem 5.1 *For any iterated securitization structure $\Sigma$, there is a tranching $\mathcal{T}$ such that*

$$p_\Sigma(r, x) = q_{\mathcal{T}}(x)$$

*for $x \in X$*

Proof: See appendix.

Theorem 5.1 states the equivalence between the equilibrium concepts in definitions 3.1 and 5.1. For any iterated securitization equilibrium, the price of the long-term asset is the same as in a waterfall equilibrium for a suitable tranching $\mathcal{T}$. Intuitively, multiple rounds of securitization are redundant since the same outcome would be obtained by tranching the long-term asset into $|\mathbb{Z}|$ securities in a single round.

## 6. EFFECT ON RATES OF RETURN

This section studies the effect of securitzation on the excess return obtained by each theory in the market, defined as the actual payoff from forgoing one bond to buy asset backed securities whenever this is perceived as worth.

The effect of securitization on returns is subtle. Agents who are not optimistic on the overall expected payoff from an underlying asset could be the most optimistic regarding the payoff being above a certain threshold. Without securitization, they will not buy the asset. By introducing the appropriate tranching, they will buy a security with an attachment point at the threshold. However, this means they will be buying the leftovers from whoever would have bought the overall asset without securitization. On the other hand, by being outbided by less sophisticated agents, they could be getting rid of tranches that do not have a promising payoff. Except for particular cases, the effect of securitization on returns tends to be ambiguous.

Formally, consider the strategy of holding one bond and exchanging it for asset backed securities only when $\mathcal{F} \in \mathbf{C}$ perceives they are worth. Denote by $\Delta_{\mathcal{F}}$ the return of this strategy in excess of always holding a bond. Let $\psi_\tau := \phi_\tau(q_{\mathcal{T}})$ be the function that maps states of the world into equilibrium payoffs for tranche $\tau \in \mathcal{T}$. Denote by $\mathbf{B}(\tau, x)$ the set of theories that buy tranche $\tau \in \mathcal{T}$ at state $x \in X$. Namely

$$\mathbf{B}(\tau, x) = \{\mathcal{F} \in \mathbf{C} : E_{\mathcal{F}}(\psi_\tau)(x) \geq E_{\mathcal{G}}(\psi_\tau)(x) \text{ for } \mathcal{G} \in \mathbf{C}\}$$

The *actual* as opposed to the *perceived* payoff from buying the security in tranche $\tau \in \mathcal{T}$ at state $x \in X$ is given by

$$T(\psi_\tau)(x) := \sum_{y \in X} \psi_\tau(y) P(x, y)$$

Finally, let $p(\tau,\cdot)$ be the equilibrium price function for tranche $\tau \in \mathcal{T}$. The excess return from buying asset backed securities whenever $\mathcal{F} \in \mathbf{C}$ perceives them as worth is

$$\Delta_{\mathcal{F}} = \sum_{x\in X}\left(\sum_{\tau\in\mathcal{T}} \left(T(\psi_\tau)(x) - R(x)p(\tau,x)\right) I_{\{\mathcal{F}\in\mathbf{B}(\tau,x)\}}\right)\mu(x) \tag{6.1}$$

It is worth remembering that, in equilibrium, every agent *perceives* that their excess return is zero. If it where positive, their demand for the long-term asset would exceed the supply. If it were negative, the agent would be buying the short-term bond instead of buying the long-term asset. Still, the *actual* average excess return $\Delta_{\mathcal{F}}^{\mathcal{T}}$ is not necessarily zero[3].

Equation (6.1) could be rewritten as the average gap between actual and perceived payoffs from each tranche. Since the equilibrium price for tranche $\tau$ is given by

$$p(\tau,x) = \frac{\max_{\mathcal{F}\in\mathbf{C}} E_{\mathcal{F}}(\psi_\tau)(x)}{R(x)}$$

it follows that

$$\Delta_{\mathcal{F}} = \sum_{x\in X}\left(\sum_{\tau\in\mathcal{T}} \left(T(\psi_\tau)(x) - E_{\mathcal{F}}(\psi_\tau)(x)\right) I_{\{\mathcal{F}\in\mathbf{B}(\tau,x)\}}\right)\mu(x)$$

meaning that the excess return is an average on the gap between actual and expected payoffs whenever the tranches are bought.

An immediate result obtained from the statistical consistency of beliefs is that if $\mathbf{C} = \{\mathcal{F}\}$, then $\Delta_{\mathcal{F}} = 0$: an agent able to match the long-run frequency of the payoff from the asset backed securities will get a return as good as holding the short-term bond as long as there are no competing theories.

When there are competing theories, the return received could be above or below the market interest rate. With no securitization, Eyster and Piccione (2013) show that, for the case where there is a theory that refines any other, no agent can earn a return above the interest rate. What is true for the whole underlying asset is still true when it is sliced into asset backed securities. To show this, assume that $\mathcal{G} \in \mathbf{C}$ refines every $\mathcal{F} \in \mathbf{C}$. Define

$$\delta_{\mathcal{F}}(\tau) := \sum_{x\in X} \left(T(\psi_\tau)(x) - E_{\mathcal{F}}(\psi_\tau)(x)\right)\left(I_{\{\mathcal{F}\in\mathbf{B}(\tau,x)\}}\right)\mu(x)$$

[3] In fact, the actual excess returns could even be positive: see Example 5 in Eyster and Piccione (2013).

as the average excess return from buying tranche $\tau \in \mathcal{T}$ on the states that $\mathcal{F} \in \mathbf{C}$ perceives this is worth. Proposition 3 from Eyster and Piccione (2013) implies that $\delta_{\mathcal{G}}(\tau) = 0$ and $\delta_{\mathcal{F}}(\tau) \leq 0$ for every $\tau \in \mathcal{T}$. Then trivailly $\Delta_{\mathcal{G}} = 0$ and $\Delta_{\mathcal{F}} \leq 0$. Whenever there exists a theory that refines every other, the excess return is invariably zero for the finest theory and non-positive for the rest, no matter the tranching.

In light of this result, it is no surprise that securitization could weakly decrease the return received by every agent, as occurs in section 2 if $\mathcal{A} = \{\{l, m, h\}\}$ and $\mathcal{B} = \{\{l\}, \{m\}, \{h\}\}$. At states where, without securitization, the most sophisticated agent would have bought the whole asset, the tranches not bought by this agent will yield a return that, on average, is below the market interest rate.

Nevertheless, securitization *could* increase the return of some agents as long as they are not the most sophisticated ones. As an example, consider a state-space $X = \{l, m, h\}$ where the probability of a state going back to itself is close to one and uniform otherwise. The collection of theories is $\mathbf{C} = \{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$ where

$$\begin{aligned}
\mathcal{A} &= \{\{l\}, \{m, h\}\} \\
\mathcal{B} &= \{\{l, h\}, \{m\}\} \\
\mathcal{C} &= \{\{l, m, h\}\}
\end{aligned}$$

The interest rate is constant and the dividend is such that, in the benchmark equilibrium, $q(l) < q(m) < q(h)$.

TABLE III

EXPECT PAYOFFS FROM THE LONG-TERM ASSET AS IN THE BENCHMARK

| States | $T(\psi)$ | $E_{\mathcal{A}}(\psi)$ | $E_{\mathcal{B}}(\psi)$ | $E_{\mathcal{C}}(\psi)$ | $\mathbf{B}$ |
|---|---|---|---|---|---|
| $l$ | $q(l)$ | $q(l)$ | $\frac{1}{2}(q(l) + q(h))$ | $\frac{1}{3}(q(l) + q(m) + q(h))$ | $\{\mathcal{B}\}$ |
| $m$ | $q(m)$ | $\frac{1}{2}(q(m) + q(h))$ | $q(m)$ | $\frac{1}{3}(q(l) + q(m) + q(h))$ | $\{\mathcal{A}\}$ |
| $h$ | $q(h)$ | $\frac{1}{2}(q(m) + q(h))$ | $\frac{1}{2}(q(l) + q(h))$ | $\frac{1}{3}(q(l) + q(m) + q(h))$ | $\{\mathcal{A}\}$ |

Consider the case without securitization. Table III presents i) the true expected price, ii) the expected price by each agent and, given these, iii) the set of agents who will buy the long-term asset. All these objects are functions of the state of the world. If the dividend at state $h$ is high enough, the asset is bought by $\mathcal{B}$ at state $l$ and by $\mathcal{A}$ at states $m$ and

$h$. Agent $\mathcal{C}$, who is less sophisticated than the other two, has a loser blessing that allows him to have a non-negative excess return:

$$\underline{\Delta}_{\mathcal{A}} = -\frac{1}{6}\left(q(h) - q(m)\right) + \frac{1}{6}\left(q(h) - q(m)\right) = 0$$

$$\underline{\Delta}_{\mathcal{B}} = -\frac{1}{6}\left(q(h) - q(l)\right) < 0$$

$$\underline{\Delta}_{\mathcal{C}} = 0$$

Consider now the case with securitization. A collection of three tranches with attachment points 0, $q(l)$ and $q(m)$ is an issuer-optimal tranching if (recycling notation) $q$ is its waterfall equilibrium and $q(l) < q(m) < q(h)$. The most senior tranche pays back $q(l)$ at every state so there is no excess return from it. The excess returns for the mezzanine and the junior tranches can be obtained from Tables IV and V, respectively. They present i) the actual expected payoffs, ii) the expected payoff for each agent and, given these, iii) the set of buyers for each of these tranches.

TABLE IV

Expect payoffs from the mezzanine tranche

| States | $T(\psi_m)$ | $E_{\mathcal{A}}(\psi_m)$ | $E_{\mathcal{B}}(\psi_m)$ | $E_{\mathcal{C}}(\psi_m)$ | $\mathbf{B}(m,\cdot)$ |
|---|---|---|---|---|---|
| $l$ | 0 | 0 | $\frac{1}{2}\left(q(m) - q(l)\right)$ | $\frac{2}{3}\left(q(m) - q(l)\right)$ | $\{\mathcal{C}\}$ |
| $m$ | $q(m) - q(l)$ | $q(m) - q(l)$ | $q(m) - q(l)$ | $\frac{2}{3}\left(q(m) - q(l)\right)$ | $\{\mathcal{A}\}$ |
| $h$ | $q(m) - q(l)$ | $q(m) - q(l)$ | $\frac{1}{2}\left(q(m) - q(l)\right)$ | $\frac{2}{3}\left(q(m) - q(l)\right)$ | $\{\mathcal{A}\}$ |

TABLE V

Expect payoffs from the junior tranche

| States | $T(\psi_j)$ | $E_{\mathcal{A}}(\psi_j)$ | $E_{\mathcal{B}}(\psi_j)$ | $E_{\mathcal{C}}(\psi_j)$ | $\mathbf{B}(j,\cdot)$ |
|---|---|---|---|---|---|
| $l$ | 0 | 0 | $\frac{1}{2}\left(q(h) - q(m)\right)$ | $\frac{1}{3}\left(q(h) - q(m)\right)$ | $\{\mathcal{B}\}$ |
| $m$ | 0 | $\frac{1}{2}\left(q(h) - q(m)\right)$ | 0 | $\frac{1}{3}\left(q(h) - q(m)\right)$ | $\{\mathcal{A}\}$ |
| $h$ | $q(h) - q(m)$ | $\frac{1}{2}\left(q(h) - q(m)\right)$ | $\frac{1}{2}\left(q(h) - q(m)\right)$ | $\frac{1}{3}\left(q(h) - q(m)\right)$ | $\{\mathcal{A}\}$ |

Thanks to securitization, $\mathcal{C}$ is lured into buying the mezzanine tranche at state $l$. As a consequence, $\mathcal{C}$'s return with securitization is lower than without. Meanwhile, at the expense of agent $\mathcal{C}$, the excess return for $\mathcal{B}$ is higher than what it would be without securitization:

$$\Delta_{\mathcal{A}} = 0 = \underline{\Delta}_{\mathcal{A}}$$

$$\Delta_{\mathcal{B}} = -\frac{1}{6}\left(q(h) - q(m)\right) > \underline{\Delta}_{\mathcal{B}}$$

$$\Delta_{\mathcal{C}} = -\frac{2}{9}\left(q(m) - q(l)\right) < \underline{\Delta}_{\mathcal{C}}$$

Thanks to securitization, some agents could get rid of toxic assets and increase their portfolio return.

## 7. OTHER TYPES OF CONTRACTS

The tranchings considered so far are constrained to the set of contracts with a waterfall structure. Still, agents could exploit even further the divergence of beliefs by creating contracts which do not satisfy this constraint. For instance, consider the equilibrium of subsection 2.2. The senior tranche pays $q(m)$ if the state next period is $m$ or $h$ and $q(l)$ otherwise, At state $m$, this tranche is bought by agent $\mathcal{B}$. This agent could create a new security, backed by the senior tranche, which pays $q(m)$ at state $h$ and 0 otherwise. Such a security does not satisfy the waterfall constraint since its payoff does not exclusively depend on the cash flow from the underlying tranche. This new security is (almost) worthless for $\mathcal{B}$, given that a realization $h$ next period looks very unlikely. However, agent $\mathcal{A}$ would be willing to pay $\frac{1}{3}\frac{q(m)}{R}$ for it.

As an alternative to the waterfall equilibrium, consider the following asset backed securities: for $y \in X$, the $y$-security pays $q(y)$ next period if the realized state is $y$ and 0 otherwise. The equilibrium price function for the long-term asset, denoted by $\bar{q}$, will be given by

$$\bar{q}(x) = \frac{d(x) + \sum_{y\in X} \bar{q}(y) \max_{\mathcal{F}\in\mathbf{C}} Q_{\mathcal{F}}(x,y)}{R(x)} \qquad (7.1)$$

for $x \in X$. It is easy to show that there exists such an equilibrium and that $q_{\mathcal{T}}(x) \leq \bar{q}(x)$

for all $x \in X$ and any tranching $\mathcal{T}$[4]. It follows that the lack of a belief that stochasticly dominates all others in at least one state is a necessary condition for tranching to increase the asset price only because of the waterfall constraint. Instead, this new kind of securitization structure will increase the price as long as there is any disagreement regarding the transition probabilities.

Apart from having their real-life counterparts, the only appealing features from securities with a waterfall structure are i) their payoff is monotone on the cash flow from the underlying asset ii) as shown in section 5, they implicitly represent any case where only debt and equity can be issued and debt itself can be used as collateral. However, no argument has been given here for agents to choose or be constrained by this class of contracts.

## 8. CONCLUSION

This paper has shown the effect of securitization under heterogeneous beliefs and short-selling constraints. The existence of this heterogeneity can only happen as long as there is bounded rationality: If rational expectations were assumed, all information (including prices) would be used efficiently and no divergence of expectations could happen. Securitization would be innocuous and asset prices will match their fundamental values. Instead, if some agents have bounded rationality, not only asset prices will be greater than their fundamental values: there could be a tranching such that everyone perceives that the underlying asset is traded at a price higher than they think it is worth.

The whole setup is built upon on the assumption that there is an infinite supply of short-term bonds. This allows agents to rollover their debt indefinitely. Since some agents might be persistently having a return below the market, their debt could increase boundlessly. Ruling this out would require a general equilibrium model which endogenizes short-term debt and accounts for wealth dynamics[5].

[4] Allen and Gale (1988) also show that, under absent short selling, the market value of an asset is maximized when the securities are such that each of them get all the payoff in a single state.

[5] Chiarella and He (2001) and Blume and Easley (2006) present heterogeneous beliefs models with these features.

In a full theoretical account of the 2008 crises, the absence of sophisticated securities will not allow belief disagreement to manifest into inflated asset prices. Once the securities are introduced, asset prices would depart from fundamentals, sending some agents -not necessarily the most naive ones- into a debt spiral. Once rollover becomes impossible, the bubble bursts. A formalization of this narrative is left for future research.

## 9. APPENDIX

PROOF OF LEMMA 4.1: Since $\mathcal{S}$ is a refinement of $\mathcal{T}$, for any funtion $g : X \to \mathbb{R}_+$ and any $x \in X$, the following inequality holds:

$$\sum_{\tau \in \mathcal{T}} \max_{\mathcal{F} \in \mathbf{C}} E_{\mathcal{F}} \left( \phi_\tau \left( g \right) \right) \left( x \right) \leq \sum_{\upsilon \in \mathcal{S}} \max_{\mathcal{F} \in \mathbf{C}} E_{\mathcal{F}} \left( \phi_\upsilon \left( g \right) \right) \left( x \right)$$

Hence $\Psi_{\mathcal{T}}(g)(x) \leq \Psi_{\mathcal{S}}(g)(x)$ and so $q_{\mathcal{T}}(x) \leq \Psi_{\mathcal{S}}(q_{\mathcal{T}})(x)$.

Define $\Psi^1_{\mathcal{S}} := \Psi_{\mathcal{S}}$ and

$$\Psi^n_{\mathcal{S}} := \Psi_{\mathcal{S}}(\Psi^{n-1}_{\mathcal{S}})$$

for $n = 2, 3, \ldots$ . Since $\Psi_{\mathcal{S}}$ is monotone (see proof of Proposition 3.1), $q_{\mathcal{T}}(x) \leq \Psi_{\mathcal{S}}(g)(x)$ for all $g$ such that $q_{\mathcal{T}}(x) \leq g(x)$ for $x \in X$. Hence if $q_{\mathcal{T}}(x) \leq \Psi^{n-1}_{\mathcal{S}}(q_{\mathcal{T}})(x)$, then $q_{\mathcal{T}}(x) \leq \Psi^n_{\mathcal{S}}(q_{\mathcal{T}})(x)$. Since $\Psi_{\mathcal{S}}$ is a contraction and $q_{\mathcal{S}}$ is its fixed point, this implies $q_{\mathcal{T}}(x) \leq q_{\mathcal{S}}(x)$ for all $x \in X$.

*Q.E.D.*

Definition 9.1, lemma 9.1 and propositions 9.1 and 9.2 will be used to prove Theorem 4.1.

DEFINITION 9.1 *A tranching $\mathcal{T}$ satisfies the dominance condition if, for each $\tau \in \mathcal{T}$ and $x \in X$, there is a theory $\mathcal{G} \in \mathbf{C}$ such that*

$$G^{\mathcal{T}}_{\mathcal{G}} \left( t|x \right) \leq G^{\mathcal{T}}_{\mathcal{F}} \left( t|x \right)$$

*for all $t \in \tau$ and all $\mathcal{F} \in \mathbf{C}$.*

LEMMA 9.1 *If $\mathcal{T}$ satisfies the dominance condition and $\mathcal{S}$ is refinement of $\mathcal{T}$, then* $q_{\mathcal{S}} = q_{\mathcal{T}}$

PROOF: For any price function $q$, the payoff from a bounded tranche $\tau$ expected for next period by theory $\mathcal{F}$ on state $x$ can be written as

$$E_{\mathcal{F}}\left(\phi_{\tau}\left(q\right)\right)\left(x\right) = \int_{\inf(\tau)}^{\sup(\tau)} \phi_{\tau}\left(t\right) dG_{\mathcal{F}}^{\mathcal{T}}\left(t|\, x\right) + \int_{\sup(\tau)}^{\infty} \phi_{\tau}\left(t\right) dG_{\mathcal{F}}^{\mathcal{T}}\left(t|\, x\right) \tag{9.1}$$

$$= \sup(\tau) - \inf(\tau) - \int_{\inf(\tau)}^{\sup(\tau)} G_{\mathcal{F}}^{\mathcal{T}}\left(t|\, x\right) d\phi_{\tau}(t) \tag{9.2}$$

Therefore, if $\mathcal{T}$ satisfies the dominance condition and $\mathcal{S}$ is a refinement of $\mathcal{T}$, then

$$\begin{aligned}\sum_{\upsilon\in\mathcal{S}:\upsilon\subset\tau} \max_{\mathcal{F}\in\mathbf{C}} E_{\mathcal{F}}\left(\phi_{\upsilon}\left(q_{\mathcal{T}}\right)\right)\left(x\right) &= \max_{\mathcal{F}\in\mathbf{C}} E_{\mathcal{F}}\left(\sum_{\upsilon\in\mathcal{S}:\upsilon\subset\tau} \phi_{\upsilon}\left(q_{\mathcal{T}}\right)\right)\left(x\right) \\ &= \max_{\mathcal{F}\in\mathbf{C}} E_{\mathcal{F}}\left(\phi_{\tau}\left(q_{\mathcal{T}}\right)\right)\left(x\right)\end{aligned}$$

for any $\tau \in \mathcal{T}$ and $x \in X$. Unsing equations (3.2) and (3.5), it follows that

$$\begin{aligned}\Psi_{\mathcal{S}}\left(q_{\mathcal{T}}\right) &= \Psi_{\mathcal{T}}\left(q_{\mathcal{T}}\right) \\ &= q_{\mathcal{T}}\end{aligned}$$

*Q.E.D.*

PROPOSITION 9.1 *If $\mathcal{T}$ satisfies the dominance condition, then*

$$q_{\mathcal{S}}(x) \leq q_{\mathcal{T}}(x)$$

*for all $x \in X$ and any tranching $\mathcal{S}$.*

PROOF: Let $\mathcal{S} \vee \mathcal{T}$ the coarsest common refinement between $\mathcal{T}$ and $\mathcal{S}$. By Lemma 4.1,

$$q_{\mathcal{S}}(x) \leq q_{\mathcal{S}\vee\mathcal{T}}(x)$$

for $x \in X$. By Lemma 9.1,

$$q_{\mathcal{S}\vee\mathcal{T}}(x) = q_{\mathcal{T}}(x)$$

for $x \in X$. *Q.E.D.*

Proposition 9.2 *If $\mathcal{T}$ does not satisfy the dominance condition, then there is a tranching $\mathcal{S}$ such that*

$$q_{\mathcal{T}}(x) < q_{\mathcal{S}}(x)$$

*for all $x \in X$.*

Proof: Since $\mathcal{T}$ does not satisfy the dominance condition, there is a state $x \in X$ and a tranche $\tau \in \mathcal{T}$ such that

$$\begin{aligned}\min_{\mathcal{F}\in\mathbf{C}} \int_{\inf(\tau)}^{\sup(\tau)} G_{\mathcal{F}}^{\mathcal{T}}(t|\,x)d\phi_\tau(t) \quad > & \quad \min_{\mathcal{F}\in\mathbf{C}} \int_{\inf(\tau)}^{\inf(\upsilon)} G_{\mathcal{F}}^{\mathcal{T}}(t|\,x)d\phi_\tau(t) \\ & + \min_{\mathcal{F}\in\mathbf{C}} \int_{\inf(\upsilon)}^{\sup(\upsilon)} G_{\mathcal{F}}^{\mathcal{T}}(t|\,x)d\phi_\tau(t) \\ & + \min_{\mathcal{F}\in\mathbf{C}} \int_{\sup(\upsilon)}^{\sup(\tau)} G_{\mathcal{F}}^{\mathcal{T}}(t|\,x)d\phi_\tau(t)\end{aligned}$$

for some $\upsilon \subset \tau$. Let $\mathcal{S}$ be a refinement of $\mathcal{T}$ such that $\upsilon \in \mathcal{S}$. From equations (3.5) and (9.2), it then follows that

$$\Psi_{\mathcal{T}}(q_{\mathcal{T}})(x) < \Psi_{\mathcal{S}}(q_{\mathcal{T}})(x)$$

Hence $q_{\mathcal{T}}$ is not a waterfall equilibrium for tranching $\mathcal{S}$. But since $\mathcal{S}$ is a refinement of $\mathcal{T}$, Lemma 4.1 implies

$$q_{\mathcal{T}}(y) \le q_{\mathcal{S}}(y) \tag{9.3}$$

for all $y \in X$ with strict inequality for $y = x$. Since for every theory the perceived probability to transit from any state to any other is always positive, the inequality is strict for every $y \in X$. *Q.E.D.*

Proof of Theorem 4.1: Follows from definition 9.1 and propositions 9.1 and 9.2. *Q.E.D.*

Proof of Theorem 5.1: From equations (5.3), (5.4) and (5.5), it follows that

$$p_\Sigma(r,x) = \frac{d(x)}{R(x)} + \sum_{z\in\mathbb{Z}} \frac{\max_{\mathcal{F}\in\mathbf{C}} E_{\mathcal{F}}(\gamma_z)(x)}{R(x)} \tag{9.4}$$

for $x \in X$.

Let $b(z)$ be the branch that connects $r$ with $z \in \mathbb{Z}$. Since $\gamma_r = p_\Sigma(r,\cdot)$, equations (5.1) and (5.2) imply

$$\gamma_z(x) = \left(p_\Sigma(r,x) - \sum_{n \in b(z)} \inf(\tau_n) - \inf(\tau_z)\right)^+ - \left(p_\Sigma(r,x) - \sum_{n \in b(z)} \inf(\tau_n) - \sup(\tau_z)\right)^+ \quad (9.5)$$

for $x \in X$.

By replacing (9.5) into (9.4) and using the equilibrium concept in Definition 3.1, it follows that $p_\Sigma(r,\cdot) = q_{\mathcal{T}}$ if the attachment points for tranching $\mathcal{T}$ are given by

$$\left(\sum_{n \in b(z)} \inf(\tau_n) - \inf(\tau_z)\right)_{z \in \mathbb{Z}}$$

*Q.E.D.*

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