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A theoretical approach to volatility surfaces in the Colombian market using the jump-diffusion model †

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Abstract

Option markets recognize that the Black & Scholes model does not account for the empirical behavior of prices. The volatility surface is the main result of such shortcoming and provides market practitioners with useful information regarding the underlying's volatility. Colombia's option market is almost inexistent and no volatility surface can be observed or calculated. In an attempt to lay down theoretical foundations for the local market, this paper approaches the volatility surface based on the jump-diffusion model. Results are not only intuitive and supported by developed market's evidence, but useful for immature options markets' development and for risk management.

JEL classification codes: G12, G13, C15.

Keywords: volatility surface, volatility smile, jump-diffusion, Brownian motion, Black & Scholes.

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1. Introduction

It is intrinsic to human nature trying to simplify reality in order to make it comprehensible. Finance and Economics are not the exception. Based on a series of assumptions, which may be unrealistic or even counterintuitive, all models are just approximations to reality and their goal is to ease comprehension and diffusion of knowledge.

Perhaps the most used assumption in Economics and Finance is the one that points out that assets' price behavior follows a Brownian motion or random walk process. The most successful and widespread models and theories are based to some extent on this assumption: the Black & Scholes (B&S) option pricing model, the CAPM asset pricing model, Markowitz's modern portfolio theory and the efficient market hypothesis, are just some examples.

Nevertheless, reality has forced market practitioners to adjust theoretical models, even if this adjustment implies the violation of model's foundations. Regarding the B&S model, market practitioners implicitly assign a different volatility for different strike prices and for different time-to-maturities, a practice which results in what is called the volatility smile and the volatility term structure, respectively. Together, the volatility smile and term structure result in the volatility surface, which is recognized as the main result of Brownian motion's flaws.

This paper pursues three main objectives. The first is to comprehend the origins and itinerary of the Brownian motion assumption in Finance and Economics. The second is to introduce the methods by which market practitioners adjust B&S model in order to circumvent the Brownian motion assumption, resulting in the mentioned volatility surface. Finally, taking into account the inexistence of a local option's market and the impossibility of obtaining observed volatilities or market-implied volatilities, an application of the jump-diffusion model is used to theoretically approximate the volatility surface for the Colombian exchange rate, fixed income and stock markets.

The results obtained are intuitive and supported by developed market's evidence and allow a better understanding of local prices and volatility dynamics. Additionally, the ability of the jump-diffusion model to capture the observed behavior of asset prices may be useful for enhancing traditional Brownian motion based risk management tools.

However, given the absence of a developed local option market, it is impossible to calibrate the model. Therefore, one of the main virtues of the volatility surface is lost: the opportunity to seize market's expectations for volatility. Though, to be able to model the volatility surface for an immature options' market may be the theoretical starting point required by market practitioners to offer and quote more contracts and, thus, promote local market development.

The remainder of this paper is organized as follows. Section 2 describes the B&S model and the historical path of its main assumption: the Brownian motion. Section 3 introduces the implied volatility and volatility surface concepts, and presents some stylized facts about the latter. Section 4 briefly describes and compares the classification of the most used methodologies for modeling the volatility surface. Section 5 introduces the application of the jump-diffusion model for modeling the volatility surface. Section 6 concludes and presents some final remarks about advantages and shortcomings of the model and its results.

2. Black & Scholes (B&S) model and Brownian motion

In 1973 Myron Scholes and Fisher Black published the model which today remains as the theoretical standard for option valuation. Originally developed for stock options' valuation[1], the B&S formula for European call options is the following:

$$C_{BS} = S_t N(d_1) - Ke^{-rT} N(d_2) \tag{1}$$

where

$$\begin{aligned} d_1 &= \frac{\ln(S_t / K) + (r + \sigma^2 / 2)T}{\sigma\sqrt{T}} \\ d_2 &= \frac{\ln(S_t / K) + (r - \sigma^2 / 2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \end{aligned} \tag{2}$$

$$\begin{aligned} C_{BS} &= \text{European call option price} \\ S_t &= \text{underlying's spot price} \\ K &= \text{strike price} \\ r &= \text{risk free interest rate} \\ T &= \text{time to maturity} \\ \sigma &= \text{underlying's volatility} \\ N(.) &= \text{cumulative normal density function} \end{aligned}$$

Albeit it is often found in the literature that Black and Scholes (1973) is the pioneer work on option valuation, related models were designed, used and documented decades before its appearance. Taleb (2007) and Haug and Taleb (2009) report that the B&S model is an alternative derivation of models already existing back then, with those models regarded as lacking of theoretical rigor despite being more robust and realist.

Taleb (2007) points out that the contribution of Black, Scholes and Merton is limited to improving an old option pricing formula and making it compatible with the existing Gaussian general financial equilibrium theories in order to make it acceptable to the economics establishment. However, such contribution came at a great cost: a high level of simplification and departure from reality, which makes the model elegant but unattractive to market practitioners.

Black and Scholes (1973) option pricing model assumes a set of "ideal conditions" in the market for the asset (stock) and the option:

i. The stock price follows a random walk in continuous time with a variance rate proportional to the square of the stock price. Thus the distribution of possible stock prices at the end of any finite interval is log-normal. The variance of the return on the stock is constant.
ii. The short-term interest rate is known and is constant through time.
iii. The option is "European", that is, it can only be exercised at maturity.
iv. There are no transaction costs in buying or selling the stock or the option.

[1] The B&S model was designed for non-dividend paying stock option valuation. Nevertheless, the model was continuously enhanced in order to be used with other assets. Therefore, the term stock and asset will be used indistinctively in the paper.

v. It is possible to borrow any fraction of the price of a security to buy it or to hold it, at the short-term interest rate.
vi. There are no penalties to short selling.

Some of these assumptions have been relaxed to make the model applicable to other assets besides non-dividend paying stocks or to recognize some market peculiarities. Despite being unrealistic, other assumptions remain because they support the model's theoretical foundations.

The most far-reaching of those assumptions is the first one, which altogether supposes that the asset's price dynamic follows a Brownian motion or random walk. This assumption includes, among others, four propositions:

i. Changes in prices are stationary; therefore, the characteristic properties of the process (e.g. drift and volatility) are time-invariant.
ii. Changes in prices are independent; there is no significant correlation with previous price changes.
iii. Changes in prices describe a normal distribution [$N\sim(0,t)$]; thus, the process is dominated by "ordinary" events, while "extreme" events occur infrequently.
iv. Changes in prices are continuous, without jumps.

Regarding the first proposition, there is evidence that the dispersion or volatility of prices changes trough time, thus making the processes non-stationary. Figure 1 provides evidence of the time-dependency of volatility for the Colombian exchange rate (TRM), fixed income (IDXTES) and stock (IGBC) markets, and for the U.S. stock market (S&P 500).[2]

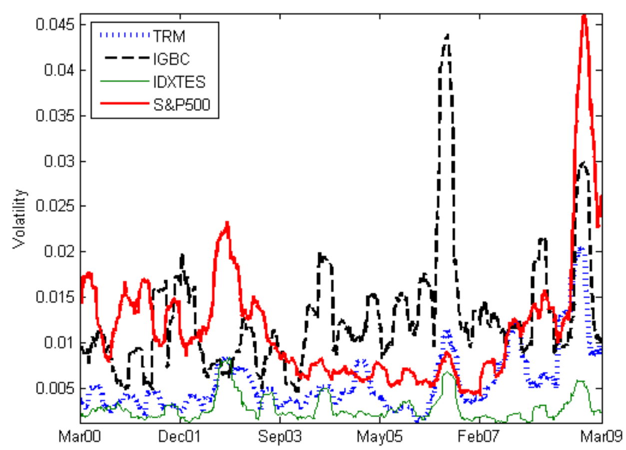


Fig.1 60-working days daily volatility

[2] TRM (Tasa Representativa del Mercado) is the local exchange rate benchmark resulting from the arithmetical average of the weighted buy and sell Peso-Dollar transactions, certified by the Colombian Banking Superintendency. IDXTES is a total return index developed by Reveiz and León (2008) for the analysis of the local public debt market. IGBC (Índice General de la Bolsa de Valores de Colombia) is the Colombian stock market index. S&P500 is the stock market index which represents approximately 75% of the U.S. stock market. Unless otherwise specified, daily time series from January 1st, 2000 to March 31st, 2009 were used for all the calculations. All calculations and figures presented in this paper used data provided by Reveiz and León (2008) and Bloomberg.

With reference to the second proposition, it is well known that big price changes tend to be followed by others of similar size, while small changes tend to be followed by others of similar size, either of positive or negative sign, in what is called volatility clustering (Mandelbrot and Hudson, 2004). Following Mikosh (2004), instead of using the standard autocorrelation plot, Figure 2 exhibits the estimated autocorrelations of the absolute values of price changes or returns. Graphically the null hypothesis of independent price changes can be rejected.[3]

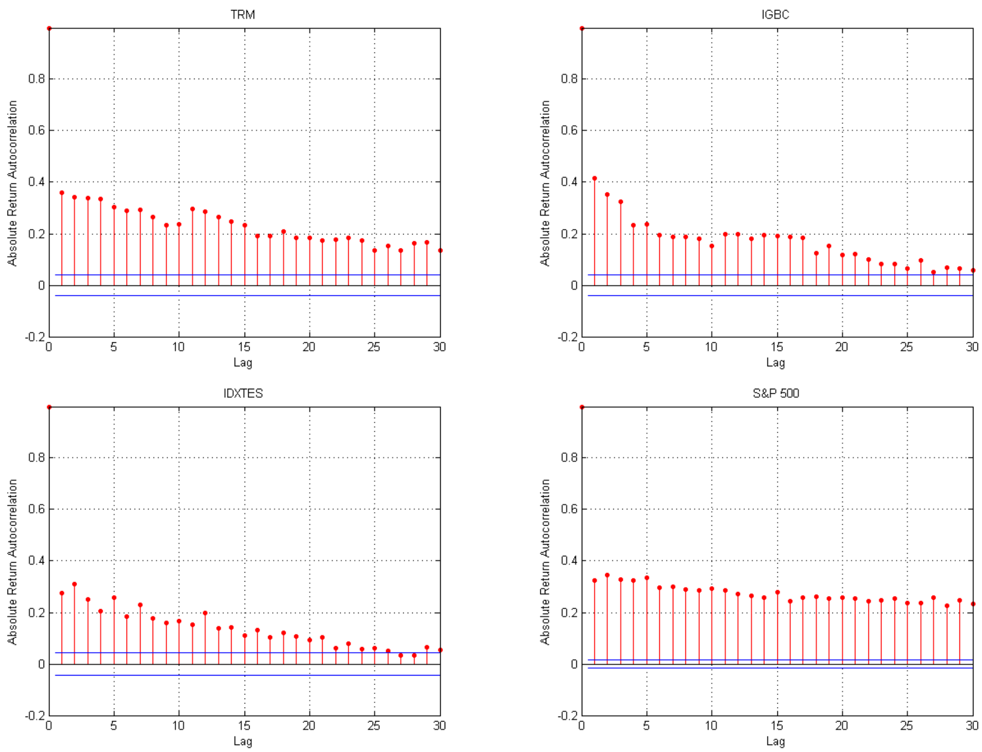


Fig.2 Autocorrelation plots

Regarding the third and fourth propositions, the normal distribution of price changes and the presence of a continuous –without jumps- process, respectively, evidence also supports their invalidity. Figure 3 exhibits the empirical standardized returns of time series and the corresponding normal benchmark; the greater the difference between the observed returns and the normal benchmark, the stronger the evidence against the normal distribution of price changes.

[3] The rejection of the independence of returns and of absolute returns null hypothesis was verified in all cases using the Ljung-Box test at the 5% confidence level.

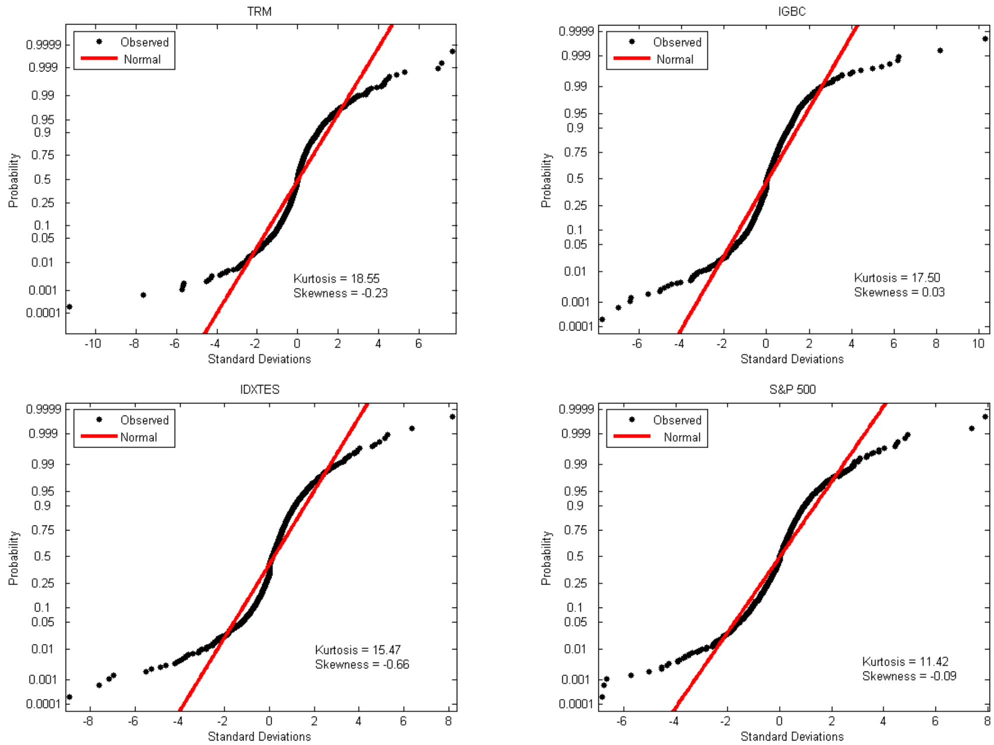


Fig.3 Probability plots

The graphic inspection confirms the departure from normal distribution of price changes.[4] The greatest divergence is located in the tails of the distribution, where the magnitude and frequency of extreme returns exceeds the normal assumption, which results in a significant excess of kurtosis or "fat tails", roughly between three and six times the corresponding kurtosis for the normal distribution. Moreover, as asserted by Malevergne and Sornette (2006), the presence of excess of kurtosis is not only a signal of excess kurtosis, but an evidence of serial correlation, which reinforces the evidence against the independence of returns proposition. Additionally, despite being dominated by the excess of kurtosis, there is also evidence of skewness, which also invalidates the normal assumption.

About the presence of a continuous process, evidence proves that price changes of unusual magnitudes do exist. Figure 4 shows the standardized returns of time series, where the presence of unusual price changes or jumps invalidate the assumption of a continuous process for price changes.

[4] The rejection of the null hypothesis of normal returns was validated with the standard tests (Jarque-Bera, Lilliefors and Kolmogorov-Smirnov).

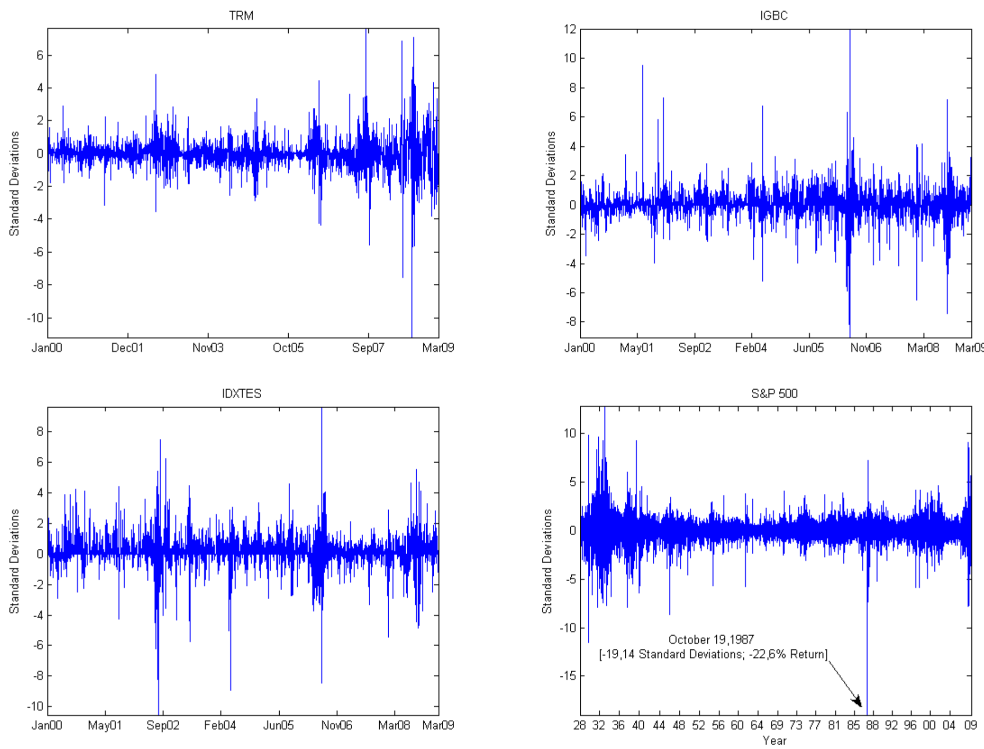


Fig.4 Standardized returns

According to the normal assumption the standardized magnitude of a price change determines its expected probability of occurrence. Under the normal assumption an event such as the one occurred in October 19th, 1987, when the U.S. stock market dropped 22.6% (-19.14 standard deviations), occurs about once each 10^{50} observations. Table 1 exhibits the expected probability of occurrence according to the normal assumption for several levels of standardized price changes, and then presents the observed incidence for each of these levels.

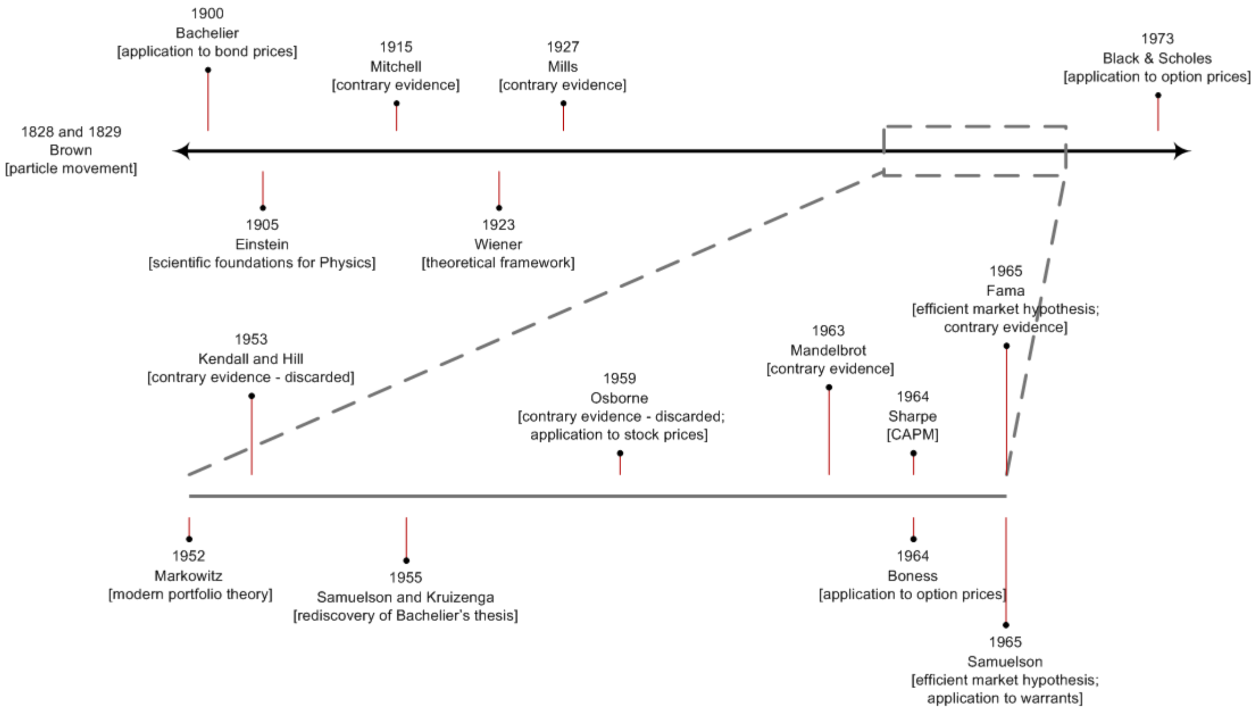
X	Normality Assumption		Observed occurrence							
			Number of occurrences				Frequency			
	Probability	One occurrence each...	TRM	IGBC	IDXTES	S&P500	TRM	IGBC	IDXTES	S&P500
>1 sigma	0,317	3 days	436	452	453	1875	18,2%	18,8%	18,9%	9,2%
>2	0,045	1 month	124	105	130	446	5,2%	4,4%	5,4%	2,2%
>3	0,0027	1.5 years	41	43	49	165	1,7%	1,8%	2,0%	0,8%
>4	6,30E-05	63 years	22	21	19	69	9,17E-03	8,75E-03	7,92E-03	3,38E-03
>5	5,70E-10	7 millenia	9	12	10	42	3,75E-03	5,00E-03	4,17E-03	2,06E-03
>6	2,00E-09	2 million years	5	8	6	23	2,08E-03	3,33E-03	2,50E-03	1,13E-03
>7	2,60E-12	1.562 million years	4	3	4	15	1,67E-03	1,25E-03	1,67E-03	7,34E-04
>8	1,20E-15	3 trillion years	1	2	2	8	4,17E-04	8,34E-04	8,34E-04	3,92E-04
>9	2,30E-19	17.721 trillion years	1	1	0	6	4,17E-04	4,17E-04	0,00E+00	2,94E-04
>10	1,50E-23	260 million trillion years	1	1	0	1	4,17E-04	4,17E-04	0,00E+00	4,89E-05

The left part of the Table presents how probable is to observe a return larger in absolute magnitude than some value equal to X times the standard deviation according to the normality assumption. The answer is given as a probability and as the number of expected occurrences per time, with a year containing 250 trading days (Sornette, 2003). The right part of the table presents the observed occurrence as number of occurrences in the sample and as a frequency; S&P calculations use daily data from 1928 to 2009, the others use daily data from 2000 to 2009.

Table 1 Expected and observed incidence of standardized returns

Therefore, as Mandelbrot and Hudson (2004) assert, financial prices do jump, and such discontinuity, far from being an anomaly best ignored, is a key element of markets that set finance apart from the natural sciences.

Despite being somewhat evident that the aforementioned propositions are flawed, the use of the Brownian motion or random walk assumption to describe the behavior of prices prevails. For that reason, to understand the origin and path of the Brownian motion is of key importance. Scheme 1 exhibits the evolution of this assumption for economics and finance, followed by a brief description of this process.



Scheme 1 Timeline of Brownian motion in Economics and Finance

Brownian motion stems from biology and physics. Its name comes from the research of Robert Brown (1828 and 1829), a Scottish botanist who described the movement of organic and inorganic particles suspended in a fluid as inexplicable, irregular and independent. Later, in the dawn of the XX century Einstein (1905) established its foundations for Physics and Weiner (1923) developed the theoretical foundations for its application.

But it was at the end of XIX century when Louis Bachelier led the way to the use of Brownian motion for describing the dynamic of asset prices. Bachelier (1900) based his theory in what he found as particularities of the dynamic of the prices of French bonds:

i. The dynamic is a "fair game", without memory, in which the probability of an increase or decrease of the prices is always ½.
ii. Probability is governed by the Law of Gauss.
iii. If time is divided in small fractions or intervals, during such interval price changes are also small.

iv. Probability is stable over time.
v. The range in which the price will be located at any time is proportional to the square root of time.
vi. The dynamic of the prices assimilates to the diffusion laws of probability of some Physic's theories, particularly the diffusion of heat.

More generally, Mandelbrot (1963) describes Bachelier theory as follows: if *Z(t)* is the price of an asset at the end of period *t*, then the successive differences of the form *Z(t+T) - Z(t)* are independent, Gaussian or normally distributed, random variables with zero mean and standard deviation proportional to the square root of the differencing interval *T*.

For decades Bachelier's application of the Brownian motion to describe the dynamic of asset prices did not transcend. It was until the 1950's when Paul Samuelson and his doctoral student R. Kruizenga rediscovered and corrected[5] Bachelier's thesis while developing a warrant valuation model (Samuelson, 1965; Samuelson, 1973; Samuelson and Crowley, 1986; Mandelbrot and Hudson, 2004). Other developments based on the application of Brownian motion to describe the behavior of prices were done by Osborne (1959) and Boness (1964). By the beginning of the 1970's, based on Boness (1964) and Samuelson (1965b), Black and Scholes (1973) developed their influential model.

But Bachelier's basics not only inspired or served option pricing theory. According to Malevergne and Sornette (2006), based on the early work of Bachelier (1900) an the improvements by Osborne (1959) and Samuelson (1965b), the log-normal paradigm has also been the staring point for theories such as Markowitz's (1952) modern portfolio theory and Sharpe's (1964) market equilibrium model (CAPM). Additionally, as asserted by Sornette (2003), Bachelier's work is also the theoretical foundation of the efficient market hypothesis developed by Samuelson (1965) and Fama (1965).

Ironically, criticism to the use of Brownian motion to describe the dynamics or behavior of prices is not new. Before Markowitz (1952), Sharpe (1964), Samuelson (1965) and Fama (1965) laid the foundations of modern financial theory several authors documented the flaws of such approach. Mitchell (1915) and Mills (1927) thoroughly documented that empirical distribution of asset prices differed significantly from the Gaussian assumption, mostly due to the presence of excess kurtosis. Later on, Kendall and Hill (1953) and Osborne (1959) coincided with Mitchell and Mills findings about the presence of excess kurtosis, but discarded those findings and justified the use of Gaussian distribution for describing the behavior of prices. Mandelbrot (1963) also found evidence of the departure from normality and proposed the use of another family of probability laws referred as "stable Paretian". Fama (1965) recognized the importance and consequences of Mandelbrot's findings, but persisted in the use of the Brownian motion assumption whilst laying the foundations of the efficient market hypothesis.

According to the literature (Fama, 1965; Belkacem *et al.*, 1996; Bhansali V. and Wise M. (2001); Mandelbrot and Hudson, 2004; Bhansali, 2005; Taleb, 2007), the flaws of Brownian motion have serious consequences for several theories or models. Markowitz modern portfolio theory, which is based on the log-normal distribution of prices and the independence of price changes, may face issues such as infinite variance; defective use of standard deviation and correlation as

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[5] According to Samuelson and Crowley (1986), Samuelson's rediscovery of Bachelier's thesis followed the advice of Professor L.J. Savage of Yale University. Samuelson not only rediscovered Bachelier´s thesis, but recognized and corrected its formulae in order to avoid prices becoming negative, in what is now called Geometric Brownian Motion, while Bachelier's is recognized as Arithmetic Brownian Motion.

volatility and dependence measures; unreliable estimation of volatility and correlation; inappropriateness of the least-squares optimization techniques; and overestimation of the benefit of diversification, among others. Regarding Sharpe's CAPM, some of the above issues may apply, resulting in a documented underestimation of the Beta coefficient about 6%. As a consequence, when applied to portfolio design and risk management, the Brownian motion assumption may be increasing the risk instead of diminishing it.

Regarding option pricing theory, the flaw of the Brownian motion results in the invalidity of the B&S model (Merton, 1976), hence the price obtained from it is erroneous. The error in the B&S model is the origin of the volatility surface.

3. The volatility surface

It is common to regard options as instruments that allow the buyer to benefit from directional movements of the market, whilst avoiding losses resulting from opposite movements (de Lara, 2002). In this sense, to buy a call (put) option allows the holder to benefit from the underlying's price increase (decrease), without facing losses in case the price decreases (increases).

However, as asserted by Malz (1998), Rebonato (1999) and Neftci (2004), from the market practitioner's point of view options are instruments of volatility; practitioners focus on the underlying's volatility, not on the direction of the price change. In fact, inside the options' market traders do not quote option prices but volatilities.

Despite contradicting the first assumption of B&S model –a constant volatility-, the market quotes different volatilities for the same underlying depending on the relation between strike price and spot price (moneyness) and the time-to-maturity of the option. Furthermore, this market practice not only contradicts B&S model, but seems counterintuitive: for a single underlying there are different volatility quotes.

Nonetheless, although contradicting B&S assumptions, the market practice of assigning different volatilities to a single underlying is nothing but the technique practitioners employ to surmount the Brownian motion flaws. As emphasized by Neftci (2004), once we accept that the use of the B&S formula amounts solely to a convention, and that traders differ in their selection of the value of the volatility parameter, then the critical process is no longer the option price, but the volatility. Therefore, one way to account for the imperfections of the B&S assumptions would be for traders to adjust the volatility parameter.

This ability of the B&S model to be adapted by market participants is what makes it so valuable, not its theoretical foundations. According to Haug and Taleb (2009), the success of B&S model its due to an attribute typical of the Gaussian distribution: you can express any probability distribution in terms of Gaussian, even if it has fat tails, by varying the standard deviation parameter [σ] at the level of the density of the random variable. For this reason, regardless of the rise of newer and better option pricing models, the market prefers to adjust B&S model instead of adopting more complex and less parsimonious models.

Consequently, despite contradicting B&S assumptions, but with the aim to preserve its simplicity and parsimony, market practitioners assign volatility as a function of moneyness [$\sigma(K,S)$] and time-to-maturity of the option [$\sigma(T)$], resulting in a volatility smile and volatility term structure, respectively. When volatility is regarded as a function of moneyness and time-to-maturity [$\sigma(K,S,T)$], a volatility surface is obtained.

Unfortunately the volatility surface [$\sigma(K,S,T)$] is directly observable in developed option markets only. This means that for other markets it should be estimated using the directly observable parameters: option price, spot price, strike price, risk free rate and time-to-maturity. This estimation process involves the backwards usage of the B&S formula, which results in the implied volatility. Therefore, implied volatility can be defined as the number that used in the B&S formula makes the theoretical price match the market price for a particular level of moneyness and time-to-maturity.

Regarding moneyness, derivatives markets define moneyness in several ways that capture the relation between the spot [S_t] and the strike price [K]. The most common moneyness metrics are the strike to spot price ratio [(K/S)], the log-return of such ratio [$ln(K/S)$], the standardized return of such ratio [$ln(K/S)/\sigma\sqrt{t}$], and the option's delta.

According to Malz (1988) and Derman (2008) market practitioners prefer using the option's delta, which corresponds to B&S' price change of the option caused by changes in the underlying's spot price. Its calculation is as follows:

$$delta_{Call} = N\left(\frac{\ln(S/K)+(r+\sigma^2/2)T}{\sigma\sqrt{T}}\right) \tag{3}$$

where

$$0 \le delta_{Call} \le 1$$
$$delta_{Call}^{ATM} \approx 0.5$$
$$delta_{Put} = delta_{Call} - 1$$

Using delta as moneyness metric has some advantages (León and Oker, 2005). Delta is bounded to values between 0 and 1 or 0 and 100[6], which implies that working with delta avoids a widening moneyness as time-to-maturity [T] increases; it is a symmetrical measure around the At The Money (ATM) level for calls and puts; it allows comparing moneyness levels between options with different underlyings; it recognizes that at certain delta levels the volatility trading is particularly liquid (e.g. 75 and 25 delta); and there exists a unique strike price for a given delta and vice versa. Additionally, according to Derman (2008), using delta is helpful since it approximates the probability that the option expires In The Money (ITM)[7].

For that reason, with delta as the metric of moneyness, Figure 5 exhibits the Bloomberg's implied volatility for the S&P 500 index on May 11th, 2009, for a five days time-to-maturity. Because B&S assumes that the volatility is independent of the moneyness level the B&S volatility should be flat; Figure 5 also presents the estimated historical volatility for the May 2008 – May 2009 and January 1928 – May 2009 periods, and verifies the departure of implied volatility from the constant volatility proposition of the Brownian motion assumption.

[6] It is a common practice to multiply the resulting delta by 100. In what is left of the document we will adopt such practice.
[7] Nevertheless, the precise probability that the option expires In The Money is not given by delta [$N(d_1)$], but [$N(d_2)$]. (Derman, 2008)

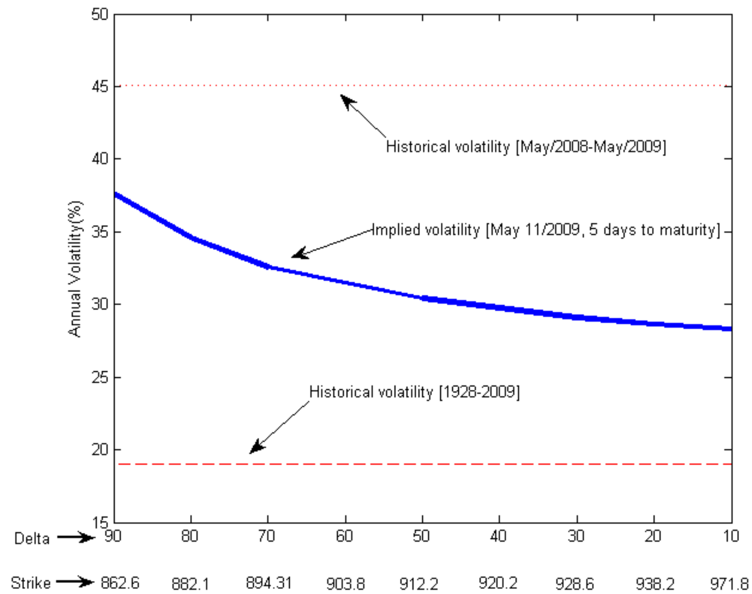

Fig. 5 S&P 500 implied volatility on May 11th, 2009, five days to maturity

Figure 5 also reveals that the implied volatility is not only flat across moneyness levels, but also not symmetrical around the center, which results in a volatility skew or volatility smirk instead of a –symmetrical- smile. This kind of volatility smile is typical of stocks or stock indexes, where market practitioners recognize fatter tails on the left side of the distribution and negative skewness.

Hull (2003) and Geman (2005) agree on two main reasons for such an asymmetric volatility smile, both related to the inability of Brownian motion to describe the behavior of prices. The first reason deals with the fact that the volatility smile appeared after the New York stock exchange crash of 1987. After this event market practitioners began to fear another extreme negative price movement, in what has been called “crashophobia”. After such event, which according to the B&S model has an extremely remote probability of occurrence, market practitioners began showing their risk aversion by relatively overpricing the volatility corresponding to sharp decreases in equity prices (León and Oker, 2005).

The second reason, documented by Black (1976) and Geman (2005), has to do with the fact that as the equity of the company declines in value, the leverage increases and also does the volatility of the equity price, making even lower stock prices more likely; vice versa, if leverage decreases, as volatility decreases, there is a lower likelihood of higher stock prices (León and Oker, 2005). Akin to this argument, Heston (1993) developed a model which considers the dependence between the underlying’s price and its volatility as relevant, thus explaining the asymmetrical volatility smile found in foreign exchange and bond markets.

According to Rebonato (1999) and Derman (2008), volatility smiles tend to exhibit a slight smirk for developed-markets exchange rates, where both currencies are deemed to be “equally powerful”. In the case of emerging markets, both authors document the existence of a more pronounced asymmetry, similar to the equity-like skew, characterized by a higher price for volatility corresponding to a depreciation of the local currency.

More generally, Derman (2004) emphasizes that each market has its own typical volatility smile shape. The shape will differ based on the fears raised by each market's bitter experiences: stock markets fear another crash, bond investors fear that interest rates increase will devalue their assets, etc.

Regarding the term structure of volatility [$\sigma(T)$], Figure 8 exhibits Bloomberg's implied volatility for ATM options on S&P 500 index on May 11th 2009. Although for this day the term structure displays little changes across time to maturities, it is not flat.

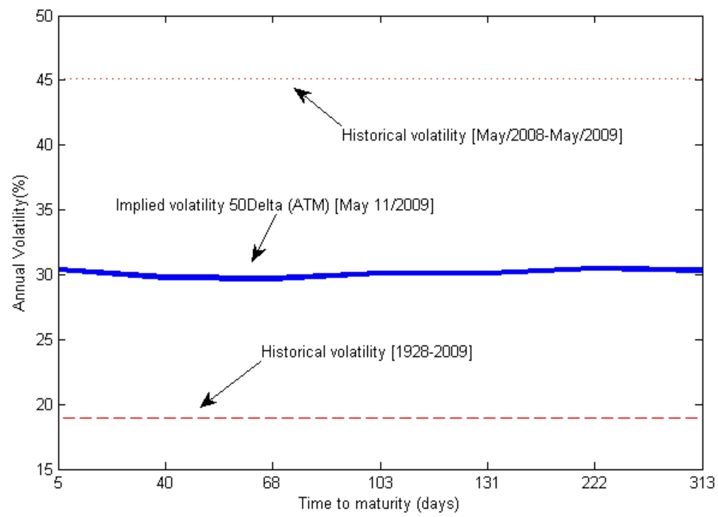

Fig. 6 S&P 500 implied volatility on May 11th, 2009, ATM options

At last, when the volatility smile and the volatility term structure concur the result is the volatility surface [$\sigma(delta, T)$].

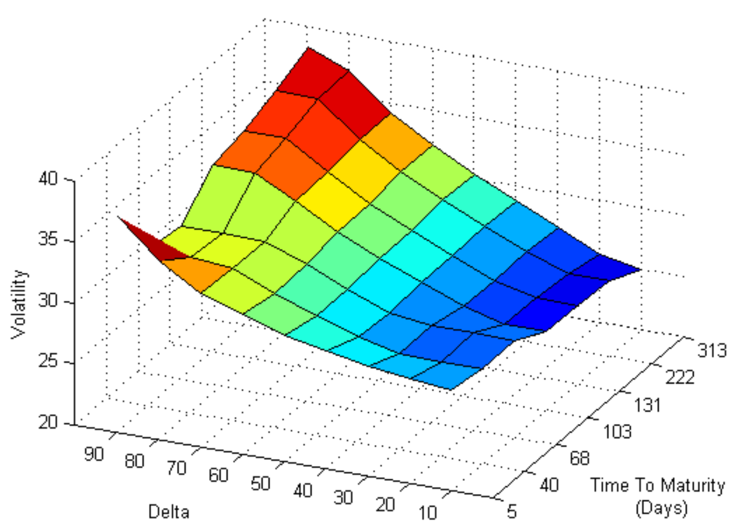

Fig. 7 S&P 500 volatility surface on May 11th, 2009

4. **Volatility surface modeling**

A volatility surface model aims at replicating the implied volatility by means of recognizing the empirical behavior of the underlying. In the literature there exist several model classifications (Duan, 1995; Rebonato, 1999; Ayache *et al.*, 2004; Derman, 2008). The main categories are (i) local volatility models; (ii) stochastic volatility models; (iii) jump-diffusion models; (iv) and universal volatility models. Table 2 describes and compares the aforementioned classification.

Name	Description	Pros	Cons
Local volatility	• First attempt to adjust the B&S model in order to match market prices. • Consists of making volatility dependant on the underlying's spot price and time to maturity. • Also known as restricted volatility models or deterministic volatility models.	• Captures the existence of the volatility surface.	• Assumes Brownian motion. • Inconsistent (flat) volatility surface for long time-to-maturities. • Useful for plain vanilla options only. • It is not a model. It does not explain the presence of the volatility surface; it just uses market prices to infer the surface.
Stochastic volatility	• Volatility is stochastic and follows a mean reversion process with its own volatility and correlation with the underlying's price. • Includes the possibility to model volatility using econometrical models (e.g. GARCH). • Also known as complete stochastic volatility models.	• Consistent for long time-to-maturities volatility surfaces; useful for pricing with long-term expirations.	• Dynamic hedge of the option via buying and selling of the underlying becomes intricate and imprecise due to issues regarding hedging the volatility of volatility. • It requires the estimation of parameters for the volatility process (mean reversion, unconditional volatility, volatility, correlation between volatility and the underlying's price). • Inconsistent (flat) volatility surface for close time-to-maturities.
Jump-diffusion	• Consists of the addition of jumps to the Brownian motion process in order to capture the empirical behavior of asset prices. • The Poisson process is used to determine the arrival of jumps.	• Consistent for short term volatility, where a prominent volatility smile does exist. • Intuitive, able to easily capture markets' discontinuities.	• Dynamic hedge of the option via buying and selling of the underlying becomes intricate and imprecise due to issues regarding discontinuities (jumps) in the process. • It requires the estimation of parameters for the jump process (magnitude and incidence). • Inconsistent (flat) volatility surface for far time-to-maturities.
Universal volatility	• It is an extension of the stochastic volatility model. • The stochastic process includes a quadratic elasticity to the change in the underlying's price. • Some versions may include jumps.	• It allows for a complete characterization of the volatility surface. • It allows estimating the slope and curvature of the surface. • It may recognize the occurrence of jumps in the process of the price.	• It requires the estimation of parameters for the stochastic volatility model and two more for the quadratic elasticity to changes in the underlying's price. • Due to its complexity and uneasy calibration, its use is limited.

Table 2 Categories of volatility surface models

However, as documented by Taleb (1997) and Derman (2004), despite the development of more advanced and sound models like those described in Table 2, market practitioners avoid new models which require the estimation of new parameters, and prefer using the implied volatility resulting from the B&S model as a convention for quoting prices in the options market.

5. A volatility surface model for the Colombian market

Colombian interest rate and stock options market is practically inexistent, whilst the foreign exchange options market is barely developed. According to BIS (2007) the only local market which traded options during 2007 was the foreign exchange market, where the OTC (Over The Counter) Colombian foreign exchange derivatives market concentrated on forward contracts (86.4% of the traded amount in 2007), with options just contributing marginally (1.6%). As a proportion of the whole foreign exchange market, options on currencies represented 2.11% of the traded amount and 1.54% of the total transactions.

In the case of the foreign exchange local options market Gómez (2008) highlights the inexistence of quotes and trades outside ATM level, the small size of quotes, and the existence of a substantial bid-ask spread. Table 3 compares the amplitude of the ATM bid-ask spreads for intraday volatility quotes on currencies in selected markets; the evidence shows that bid-ask spread in Colombia is remarkably high, even when compared to other emerging markets.

	Currency				
	Colombian Peso	Mexican Peso	Brazilian Real	Euro	Japanese Yen
Mean	323.4	239.2	188.2	57.7	86.6
Median	350.0	235.0	162.5	56.6	82.5
Standard Deviation	86.4	78.1	89.3	20.6	31.0
Maximum	700.0	625.0	675.0	164.0	226.5
Minimum	200.0	45.0	65.0	5.0	10.0

Table 3 ATM bid-ask spreads for intraday volatility quotes on currencies (basis points)
Source: Bloomberg.

Because of the Colombian options market facts aforementioned, it is appropriate to affirm that a volatility market does not exist. Therefore, it is impossible to directly observe the volatility surface or to calculate the implied volatility to indirectly observe it.

However, it is possible to estimate the theoretical volatility surface by means of simulating the stochastic process which describes the behavior of asset prices in the Colombian market. The proposed technique for such estimation begins by choosing an appropriate stochastic process to simulate via the Montecarlo method the price paths of the underlying asset. Then, using B&S formulae, the price paths are used to value options which differ in their strike price and time-to-maturity. This is akin to calculating the market implied volatility but with appropriate non-Brownian-simulated prices instead of market prices, thus the result is a theoretical volatility surface in which the B&S volatility parameter is adjusted in order to match market simulated prices.

Taking into account the advantages and disadvantages of the main volatility surface models (Table 2), but especially because of its parsimony and ability to recognize the existence of an empirical process characterized by discontinuities, this paper implements Merton's (1976) jump-diffusion model's essentials to simulate the price paths of the underlying. This choice is adequate since most market participants nowadays regard jumps as the main factor determining the very short-term volatility smile (Derman, 2004); because the jump-diffusion model is convenient due to its realism (Derman, 2008); and because of its ability to capture the existence of fat tails in the distribution of financial assets (Wilmott, 2007).
Furthermore, because the local market is practically inexistent, the choice of the jump-diffusion model is adequate since it better captures the volatility smile at short times-to-maturity, which are the most likely to develop first.

According to Gatheral (2006) and Wilmott (2007) the implementation of the jump-diffusion model for modeling the price of an asset is as follows:

$$dS = \mu S dt + \sigma S dZ + JSdq \tag{4}$$

where

$$dq = \begin{cases} 0 & \text{with probability} \quad 1-\lambda(t)dt \\ 1 & \text{with probability} \quad \lambda(t)dt \end{cases}$$

y

S = spot price

J = jump size

μ = mean of the Brownian motion process

σ = standard deviation of the Brownian motion process

λ = intensity of the Poisson process

dZ = Brownian motion process

The jump-diffusion model assumes that the behavior of price changes is characterized by two independent processes (Merton, 1976; Gatheral, 2006). The first is based on geometric Brownian motion [*dZ*] and captures the "normal" vibrations in price, which only cause marginal changes price movements. The second, based on a Poisson distribution [*dq*], captures the "abnormal" or discontinuos behavior of price changes, which affect prices in a non-marginal manner. Using Mandelbrot's (2003) terms, these two processes allow the jump-diffusion model to capture the "mild" and "wild" volatility which characterize asset prices, respectively.

The proposed Montecarlo method to simulate prices based on the jump-diffusion model consists of the simulation of 20,000 price paths with daily frequency, where the Brownian motion's volatility [*σ*] and mean [*μ*] are estimated in a standard manner, whereas the jump size [*J*] and intensity [*λ*] parameters require special attention.

In order to estimate jump size [*J*] and intensity [*λ*] parameters there must be a threshold for what is to be considered "normal" or "abnormal" changes in price. Reviewed literature on jump-diffusion models does not define a method for determining what a jump is. Therefore, the chosen approach is a convenient, yet sound and coherent one: jumps are all price changes of size above a threshold [*h*] that result in excess kurtosis in the distribution of asset prices' returns, with [*h*] being measured in terms of standard deviations [±*σ*].

The estimation of the mentioned [*h*] threshold results in two time series subsets.[8] The first subset [*Cb*], characterized by a nil excess kurtosis, contains all “normal” price changes and is used to estimate the Brownian motion’s volatility [*σ*] and mean [*μ*] parameters. The second subset [*Cj*] corresponds to all standardized price movements above the [*h*] threshold, and is used to estimate the intensity[9] [*λ*] and size [*J*] of the jump.

Regarding the size of the jump [*J*], literature assumes the log-return of the jump size to be independently and identically normally distributed, which results in the convenient closed form formula for option pricing first proposed by Merton (1976). Nevertheless, because such assumption would result in rather symmetrical volatility smiles, and since finding a closed form solution is outside of the scope of this paper, the estimation of the jump parameter [*J*] follows a non-parametric approach. Instead of making any assumption about the distribution of the jumps, the herein proposed model draws an observed jump from the second subset [*Cj*] each time a jump is generated according to the intensity parameter [*λ*].[10]

The described approach to simulate the diffusion and jump processes has two main advantages. First, since the diffusion process is based on the estimation of the volatility [*σ*] and mean [*μ*] parameters from a time series subsample [*Cb*] without excess kurtosis, it is safe to simulate it using Brownian motion. Second, the jump process is obtained using a non-parametric simulation which is capable of capturing the bitter experiences which Derman (2004) emphasizes to be the main driving factor behind the shape of the volatility smile.

The results of using the described approach are presented in Figure 8, which displays the simulated paths and the histogram for the TRM (COP/USD). The Brownian motion and jump-diffusion results are presented in different colors, where the latter exhibits extreme returns.

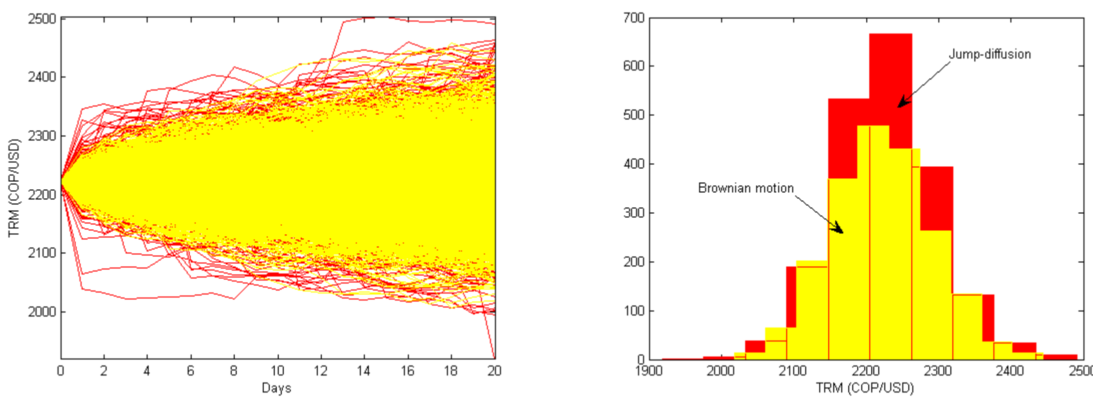

Fig. 8 TRM (COP/USD) simulated prices

[8] The estimated [*h*] threshold is ±1.36, ±1.68 and ±1.48 standard deviations for the TRM, IGBC and IDXTES, respectively; in each case the jump’s subset [*Cb*] consisted of 219, 147 and 242 observations, respectively.

[9] In order to estimate the intensity of the jump the whole time series is used in conjunction with the second subset, where the latter allows to locate jumps occurring within the former.

[10] The procedure of drawing an observation from the jumps’ subset is somewhat similar to the first stage of the bootstrap procedure described by Dowd (2005). The jump drawing consists of a random sampling, or resampling, based on a uniform random number generator to select a random number between 1 and *n*, where *n* is the size of the original sample of the jumps’ subset [*Cj*]. Some advantages of such method are higher accuracy and precision of the estimates, and avoiding strong assumptions about the relevant distribution of the data or requiring large samples.

Figure 9 displays the probability plots of the jump-diffusion simulated returns and of observed returns. If compared with the Brownian motion model (Figure 3), Figure 9 exhibits a noteworthy improvement in the ability of the model to fit empirical data.

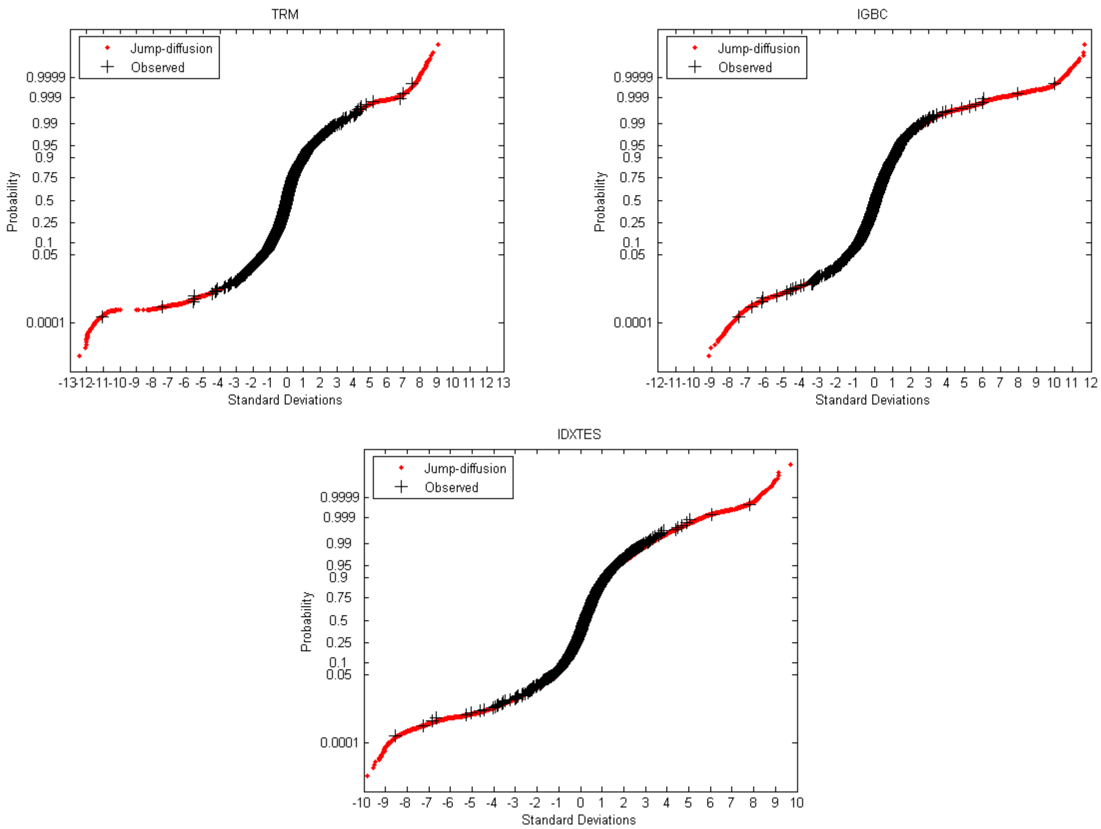

Fig. 9 Probability plots

Finally, using B&S backwards[11], jump-diffusion model simulated prices resulted in the theoretical volatility surfaces presented in Figure 10.

[11] In the TRM case the B&S model was adjusted as proposed by Garman and Kohlhagen (1983) for currency options.

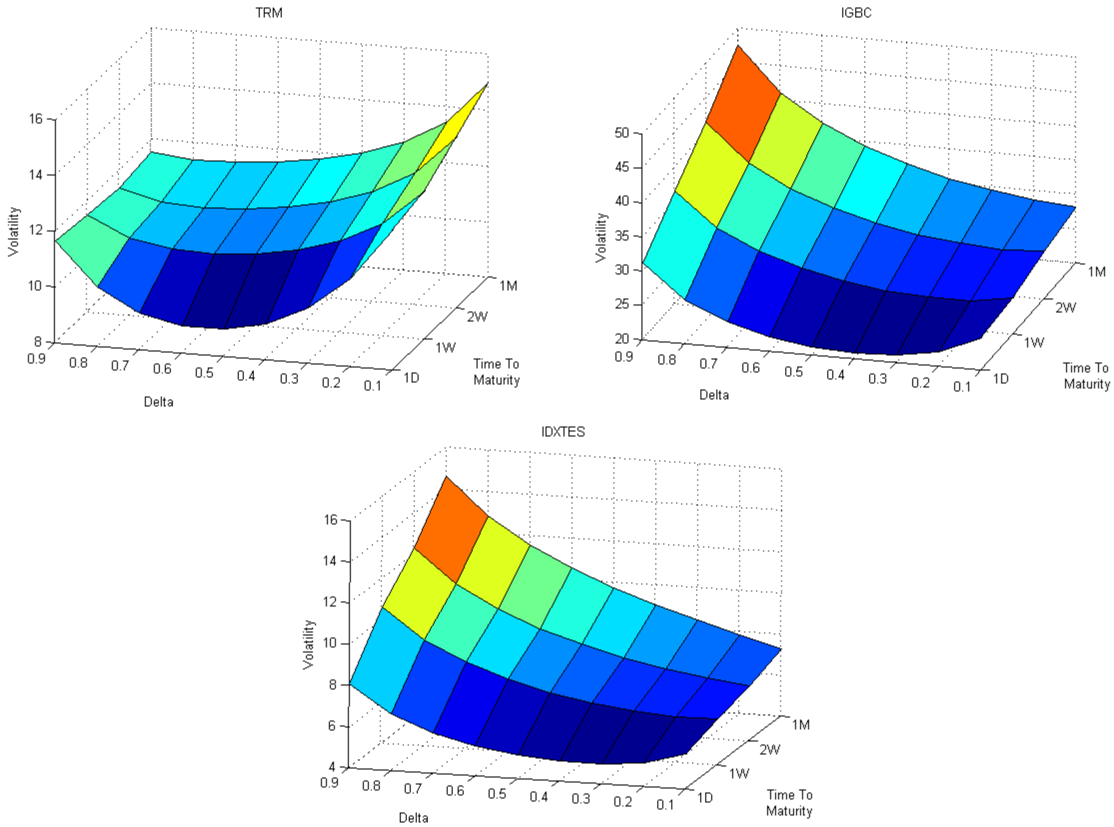

Fig. 10 Volatility surfaces

The resulting volatility surfaces are intuitive, supported by theory and other market's markets experience. In the case of TRM, because Colombian Peso is considered a weak currency against U.S. Dollar, market participants should assign a higher volatility to scenarios consisting of steep depreciations of local currency. The IGBC case exhibits the typical "crashophobia" and leverage effect previously discussed, which results in higher volatility to scenarios where sharp decreases of the stock market occur. Regarding IDXTES index, market participants should assign a higher volatility to those states in which interest rate increments result in sharp depreciations of the local public debt market bonds.

6. Final remarks

Despite not being supported by empirical observation, the B&S model has withstood the test of time. As documented in this paper, B&S value does not reside in its theoretical foundations, but in its ability to accommodate market practitioner's adjustments to properly price options. This ability makes B&S the market's convention for quoting volatility.

Based on this ability, this paper developed a jump-diffusion Montecarlo simulation which captured the empirical behavior of financial prices in an effective and parsimonious manner, resulting in a theoretical approach to the modeling of volatility surfaces.

Nevertheless, it is indispensable to recognize several shortcomings of the proposal. First, despite Derman's (2004) assertion about the relevance of bitter experiences in characterizing the shape of the volatility smile, to assume the past as an indicator of all the possible estates of the nature will always be subject to criticism, and the length of the time series will always be an issue. Second, since there is no local options market, the model can not be calibrated and the resulting surface may not be used as the market expected volatility, but a theoretical benchmark for quoting volatility.

However, it is worth stressing the importance of two of the paper's contributions. First, resulting volatility surfaces are intuitive, yet coherent with theory and empirical observation. Second, as exhibited in Figure 9, the model was able to better capture the empirical behavior of market prices, even generating extreme movements not found in the time series. This fact makes the proposed jump-diffusion Montecarlo simulation method also valuable for risk management, where Value at Risk-type methods have recently demonstrated their limitations when facing extreme events.

Extensions for future research include the model's calibration as the market develops and the ability of the model not only to capture fat tails in the distribution of asset prices, but to be able to reproduce the serial dependence of returns. The former extension will make the model more valuable for local practitioners to use in their quotes and trades, whilst the latter will enhance the ability of the model to model the empirical dynamics of asset returns.

Finally, it is worth mentioning that this approach is a theoretical one, which attempts to lay down theoretical foundations for an immature options market in order to facilitate its development. Market development will allow theoretical prices to be closer to market prices, enhancing the ability of market participants, authorities and agents to take part of the derivatives market; meanwhile, this model may be one of the theoretical starting points to promote such market development.

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